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Diffusive Limit of a Two Dimensional Kinetic System of Partially Quantized Particles

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Abstract A quantum-classical coupled system which models the diffusive transport of electrons partially confined in semiconductors nanostructures was presented in Ben Abdallah and Méhats (Proc. Edinb. Math. Soc. 49:513–549, 2006). In this model, electrons are assumed to behave like wave in the confinement direction and to have a classical behaviour in a diffusive regime in the transport direction parallel to the electron gas. It was formally derived from a kinetic system for partially quantized particles thanks to a diffusive limit when the mean free path becomes small with respect to the macroscopic length scale. This paper is devoted to the rigorous study of this limit for a transport in one dimension. In the transport direction, the motion of particles is described by a 1D Boltzmann equation. A Boltzmann-Schrödinger-Poisson system is then considered. Existence of renormalized solutions relying on the study of a quasistatic Schrödinger-Poisson system and on an entropy estimate is established. Its diffusive limit is then considered.

Keywords Schrödinger equation · Boltzmann equation · Poisson equation · Entropy inequality · Subband method · Diffusion limit · Renormalized solution · Drift-diffusion equation · Semiconductors

1 Introduction and Main Results

By downscaling electronics components at nanometer scale, quantum effects become non-negligible. In nanoscale semiconductor devices, electrons might be extremely confined in one or several directions due to the length scales. These directions are referred to as the confining directions. This leads to a partial quantization of the energy. The subband decomposition approach [27, 36, 37] was introduced by several authors in order to take advantage of this reduction of dimensionality. This method consists of a separation of the confinement and the transport directions.

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In the non-confined direction(s), that we shall also refer to as the transport direction(s), transport might have a quantum nature or be purely classical in the kinetic or diffusive regimes. In this work, we are interested in the kinetic regime (the diffusive regime has been studied in [5, 34]) and in the convergence from the kinetic model to the diffusive model. One of the most used models to describe the transport of charged particles in a kinetic approach in several domain such as plasmas or semiconductors is the Boltzmann transport equation [6, 22, 31, 33].

In the confined direction, electrons behave like waves. The system is at thermodynamical equilibrium and is described by the subband model as a statistical mixture of eigenstates of the 1D stationary Schrödinger equation.

Namely, we consider a particle system of charged carriers which is partially quantized in one direction (denoted by z) and which, in the transport direction denoted by x, is in a kinetic regime. The coupling occurs then in the momentum variable. We will first briefly describe the model used and refer the reader to [26] for more details. A Vlasov-Schrödinger-Poisson system which presents also a similar quantum-classical coupling is analyzed in [4].

1.1 The Schrödinger-Poisson System

In the transverse direction (referred by z), electrons are confined in the nanostructure. The description of the system needs the diagonalization of the 1D stationary Schrödinger equation. We define then on $\Omega = (a,b) \times (0,1)$, the set $(\chi_k[V], \epsilon_k[V])_{k\geq 1}$ as the complete set of eigenfunctions and eigenvalues of the Schrödinger operator in the z variable, $z \in (0,1)$:

$$\begin{cases} -\frac{1}{2}\partial_z^2 \chi_k[V] + V\chi_k[V] = \epsilon_k[V]\chi_k[V] \quad (k \geq 1), \\ \chi_k[V](0) = \chi_k[V](1) = 0, \qquad \int_0^1 |\chi_k[V]|^2\, dz = 1. \end{cases} \tag{1.1}$$

The square of the modulus of the wave functions $(\chi_k[V])_{k\geq 1}$ represents the probability of occupation on the kth subband. If we denote ρ_k the occupation number of the kth subband, which is defined below by $\int f_k\, dv$, the particle density for a partially quantized system can be written

$$N(t,x,z) = \sum_{k=1}^{+\infty} \rho_k(t,x)|\chi_k[V(t,x,\cdot)](z)|^2.$$

The electrostatic potential V generated by the charged carriers is then the solution of the Poisson equation:

$$-\Delta_{x,z} V(t,x,z) = \sum_k \rho_k(t,x)|\chi_k[V(t,x,\cdot)](z)|^2, \tag{1.2}$$

with the boundary conditions:

$$\begin{cases} \frac{dV}{dx}(t,a,z) = \frac{dV}{dx}(t,b,z) = 0, \quad \text{for } z \in (0,1), \\ V(t,x,0) = V(t,x,1) = 0, \qquad \text{for } x \in (a,b). \end{cases} \tag{1.3}$$

The boundary conditions here are chosen such in order to simplify the mathematical analysis, moreover elliptic regularity of the Poisson equation (1.2) are needed in our proofs. However, in the spirit of [5], we can extend the proofs to the case where $V(t,x,0) = V_b^0(x)$ and $V(t,x,1) = V_b^1(x)$ with $\frac{d}{dx}V_b^0(a) = \frac{d}{dx}V_b^1(b) = 0$. The idea is to introduce the extension $\underline{V}$ on Ω of the boundary data and to consider the quantities $V - \underline{V}$ instead of V, $\epsilon_k[V] - \epsilon_k[\underline{V}]$ instead of $\epsilon_k[V], \ldots$.

The Schrödinger-Poisson system was solved in [24, 25] by variational methods. Such techniques are used here to obtain existence and uniqueness of solutions of this system for a given $\rho = (\rho_k)_{k\geq 1}$.

In the following, when there is no confusion possible, we will denote ϵ_k instead of $\epsilon_k[V]$ and χ_k instead of $\chi_k[V]$.

1.2 The Transport Equation

The Boltzmann equation is one of the most used equation describing the transport of charged carriers in semiconductors in a kinetic regime [29, 33]. Let $\eta > 0$ be the scaled mean free path assumed to be small and denote V^η the electrostatic potential generated by the charged carriers. We consider here the scaled Boltzmann equation in one dimension for the subband model defined on the phase space $(a,b)\times\mathbb{R}$. The position x belongs to (a,b), the velocity v belongs to $\mathbb{R}$ and the time variable t is nonnegative. Then the occupation number ρ_k^η is defined by $\rho_k^\eta = \int_{\mathbb{R}} f_k^\eta\, dv$ where the distribution function $f_k^\eta(t,x,v)$ satisfies

$$\partial_t f_k^\eta + \frac{1}{\eta}(v\,\partial_x f_k^\eta - \partial_x\epsilon_k[V^\eta]\,\partial_v f_k^\eta) = \frac{1}{\eta^2} Q^\eta(f^\eta)_k. \tag{1.4}$$

By using the notation $\{\cdot,\cdot\}$ for the Poisson bracket: $\{g,h\} = \partial_x h\,\partial_v g - \partial_v h\,\partial_x g$, we can rewrite the Boltzmann equation:

$$\partial_t f_k^\eta + \frac{1}{\eta}\{\mathcal{H}_k^\eta, f_k^\eta\} = \frac{1}{\eta^2} Q^\eta(f^\eta)_k,$$

where $\mathcal{H}_k$ denotes the energy of the system in the kth subband which is the sum of the kinetic energy and the potential energy:

$$\mathcal{H}_k^\eta(t,x,v) = \frac{1}{2}v^2 + \epsilon_k[V^\eta(t,x,\cdot)].$$

In semiconductors, the main mechanism driving the electrons towards a diffusive regime is collision with phonons (vibration of the semiconductor crystal lattice) [32]. The collision operator Q^η for the electron-phonon interaction in the linear BGK approximation reads in the following form:

$$Q^\eta(f)_k = \sum_{k'}\int_{\mathbb{R}} \alpha_{k,k'}(v,v')(\mathcal{M}_k^\eta(v) f_{k'}(v') - \mathcal{M}_{k'}^\eta(v') f_k(v))\,dv', \tag{1.5}$$

where the function $\mathcal{M}_k^\eta$ is the normalized Maxwellian

$$\mathcal{M}_k^\eta(t,x,v) = \frac{1}{2\pi\mathcal{Z}^\eta} e^{-\mathcal{H}_k^\eta(t,x,v)} \tag{1.6}$$

and where the repartition function $\mathcal{Z}^\eta$ is given by

$$\mathcal{Z}^\eta(t,x) = \sum_{k=1}^{+\infty} e^{-\epsilon_k[V^\eta(t,x,\cdot)]}. \tag{1.7}$$

We refer the reader to [7, 31, 33] for a physical background on the Boltzmann equation (1.4).

The equation is completed with the specular reflection boundary conditions:

$$f_k^\eta(t,a,v) = f_k^\eta(t,a,-v), \qquad f_k^\eta(t,b,v) = f_k^\eta(t,b,-v), \quad v>0, t\in\mathbb{R}^+. \tag{1.8}$$

The surface density of particles is defined by

$$N_s^\eta(t,x) = \int_0^1 N^\eta(t,x,z)\,dz = \sum_k \int_{\mathbb{R}} f_k^\eta(t,x,v)\,dv = \sum_k \rho_k^\eta(t,x).$$

The cross section α is assumed to be symmetric and bounded from above and below:

(A.1) $\alpha_{k,k'}(v,v') = \alpha_{k',k}(v',v)$ and $0<\alpha_1 \le \alpha_{k,k'}(v,v') \le \alpha_2$, for all $(v,v')\in\mathbb{R}^2, k,k'\ge 1$.

We considered the well-prepared initial condition assumed to be at the *thermal equilibrium*:

$$f_k^\eta(0,x,v) = f_k^{in}(x,v) := \frac{N_s^{in}(x)}{2\pi\sum_k e^{-\epsilon_k[V^{in}]}}\, e^{-v^2/2-\epsilon_k[V^{in}]}, \quad (x,v)\in[a,b]\times\mathbb{R}, \tag{1.9}$$

where $(V^{in}, (\epsilon_k[V^{in}], \chi_k[V^{in}])_{k\ge1})$ is the set of solutions of the Schrödinger-Poisson system at thermal equilibrium:

$$\begin{cases} -\frac{1}{2}\,\partial_z^2\chi_k[V^{in}] + V^{in}\chi_k[V^{in}] = \epsilon_k[V^{in}]\chi_k[V^{in}] \quad (k\ge1), \\ \chi_k[V^{in}](x,\cdot)\in H_0^1(0,1), \qquad \int_0^1 \chi_k[V^{in}]\,\chi_\ell[V^{in}]\,dz = \delta_{k\ell}, \end{cases}$$

$$-\Delta_{x,z}V^{in} = \sum_k \frac{N_s^{in}(x)}{\sum_k e^{-\epsilon_k[V^{in}]}}|\chi_k[V^{in}]|^2 e^{-\epsilon_k[V^{in}]}.$$

We assume that we have

(A.2) $N_s^{in}\ge 0$, $N_s^{in}\in C^0([a,b])$.

Under this assumption, it has been stated in Proposition 2.1 of [5] that the above Schrödinger-Poisson system at thermal equilibrium admits a unique set of solution $(V^{in}, (\epsilon_k[V^{in}], \chi_k[V^{in}])_{k\ge1})$ with $0\le V^{in}\in C^1(\Omega)$, where we recall that $\Omega=(a,b)\times(0,1)$.

From a mathematical point of view, the diffusive limit is obtained by letting η going to 0 in (1.4). It is well-known that in a diffusion approximation the surface density N_s satisfies at the limit a drift-diffusion equation [14, 29]. We propose here to extend these results for the coupled quantum-classical system presented above.

Before stating the results of this paper, let us introduce some notations. An originality of this system is the infinite sequence of solution of kinetic equations. Then we denote for any separable Banach space E by $\ell^1(E)$ the space of sequences $(h_k)_{k\ge1}$ such that for all $k\ge1$ we have $h_k\in E$ and $\sum_{k\ge1}\|h_k\|_E < +\infty$, this last quantity being the norm of $(h_k)_{k\ge1}$ in $\ell^1(E)$. Its dual is $\ell^\infty(E')$ the set of sequences $(u_k)_{k\ge1}$ belonging to the dual E' of E such that $\sup_k \|u_k\|_{E'}$ is finite. We say that a sequence $(h_k^n)_{k\ge1}$ converges weakly to (h_k) in $\ell^1(E)$ if for any $(u_k)_{k\ge1}\in\ell^\infty(E')$, we have $\sum_k\langle h_k^n - h_k, u_k\rangle_{E',E}\to 0$ as $n\to\infty$. We recall that as a consequence of the Dunford-Pettis Theorem and the De La Vallée Poussin Theorem, a sequence $(h^n)_n$ is relatively weakly compact in $\ell^1(L^1(\mathcal{O}))$ (for $\mathcal{O}\subset\mathbb{R}^N$) if there exists a nonnegative function G satisfying $\lim_{t\to+\infty}\frac{G(t)}{t}=+\infty$ and such that $\sup\sum_k\int_\Omega G(|f_k|)\,dx<+\infty$ (see Chap. 2 of [12]). All along the paper, we will usually shortly denote by $\|h_k\|_{L^p_{t,x,v}}$ the $L^p((0,T)\times[a,b]\times\mathbb{R})$ norm of h_k. Finally, we will make use of the space $L\log L(\mathcal{O})$ defined as the space of positive function f such that $\psi(f)\in L^1(\mathcal{O})$ where $\psi(x)=x\log x$.

1.3 Main Results

We are interested in this paper in the diffusive limit of the Boltzmann-Schrödinger-Poisson system presented before:

$$\partial_t f_k^\eta + \frac{1}{\eta}(v.\partial_x f_k^\eta - \partial_x \epsilon_k^\eta.\partial_v f_k^\eta) = \frac{1}{\eta^2} Q^\eta(f^\eta)_k, \quad (x,v) \in (a,b)\times\mathbb{R}, \tag{1.10}$$

$$\begin{cases} -\frac{1}{2}\partial_z^2 \chi_k^\eta + V^\eta \chi_k^\eta = \epsilon_k^\eta \chi_k^\eta & (k \geq 1), \\ \chi_k^\eta(t,x,\cdot) \in H_0^1(0,1), & \int_0^1 \chi_k^\eta \chi_\ell^\eta \, dz = \delta_{k\ell}, \end{cases} \tag{1.11}$$

$$-\Delta_{x,z} V^\eta = \sum_k \int_{\mathbb{R}} f_k^\eta |\chi_k^\eta|^2 \, dv, \tag{1.12}$$

which is coupled with the boundary condition (1.8) and (1.3) and the well-prepared initial boundary condition (1.9). The aim of this paper is to prove rigorously the limit as η goes to 0 of this system to the drift-diffusion-Schrödinger-Poisson system studied in [5]. One particular relevant motivation of this work is to derive a model for which numerical simulations are less costly and simpler than for the kinetic-quantum model (1.10)–(1.12). Then a numerical simulation of the drift-diffusion-Schrödinger-Poisson system obtained as η goes to 0 is provided in [26] to simulate the diffusive transport of electrons in a double-gate MOSFET. An interesting continuation of this work is to extend to more general collision operators to derive a hierarchy of classical-quantum coupled model in the spirit of [3].

To establish rigorously the diffusive limit, we will make use of techniques which have been developed in the framework of hydrodynamics limits for the Boltzmann equation by several authors (see e.g. [1, 8, 13, 15, 16, 20, 30] and see [35] for a review). Diffusion limits for parabolic systems have been presented in [9], where linear kinetic equations arising in models of plasma or semi-conductors or rarefied gases are considered, and in [19] for generalized two-velocity models.

Although the linearity of the collision operator Q, the coupling is highly non linear and then we are not able to construct strong solutions for this system. Thus we will work in the framework of renormalized solutions [10, 11, 23].

Definition 1.1 We say that a nonnegative function $f^\eta = (f_k^\eta)_{k\in\mathbb{N}^*}$ is a renormalized solution of (1.10) if $\forall \beta \in C^1(\mathbb{R}^+)$ with $|\beta(t)| \leq C(\sqrt{t}+1)$ and $|\beta'(t)| \leq C$, we have for all $k \geq 1$, $\beta(f_k^\eta)$ is a weak solution of:

$$\begin{cases} \eta\partial_t \beta(f_k^\eta) + v\,\partial_x \beta(f_k^\eta) - \partial_x \epsilon_k^\eta\,\partial_v \beta(f_k^\eta) = \frac{Q^\eta(f^\eta)_k}{\eta}\beta'(f_k^\eta), \\ \beta(f_k^\eta)(t=0) = \beta(f_k^{in}), \\ \beta(f_k^\eta)(t,a,v) = \beta(f_k^\eta)(t,a,-v), \qquad \beta(f_k^\eta)(t,b,v) = \beta(f_k^\eta)(t,b,-v), \quad v>0,\ t>0. \end{cases}$$

The entropy of the system is defined by

$$W^\eta(t) = \sum_k \iint_{(a,b)\times\mathbb{R}} \left(f_k^\eta \log\frac{f_k^\eta}{M_k} - f_k^\eta + M_k \right) dx\,dv + \frac{1}{2}\iint_\Omega |\nabla_{x,z} V^\eta|^2 \, dx\,dz, \tag{1.13}$$

where $M_k = K\exp(-\frac{1}{2}(v^2+k^2))$ with a constant K chosen such that $\sum_k \int M_k\,dv = 1$. The dissipation rate which measures the distance to the equilibrium is defined by

$$\mathcal{R}^\eta(t) = \frac{1}{2}\sum_k \iint_{(a,b)\times\mathbb{R}} \left(\sqrt{f_k^\eta} - \sqrt{N_s^\eta \mathcal{M}_k^\eta}\right)^2 dx\,dv. \tag{1.14}$$

Remark We point out the fact that, looking at the expression of the entropy of the system, we do not have better estimates in space than $L\log L$ for f^η and H^1 for V^η. It is proved in Appendix that it implies a bound of $\partial_x\epsilon_k^\eta$ in L^2. Thus the product $f_k^\eta\cdot\partial_x\epsilon_k^\eta$ has no meaning even in a weak sense. The renormalization of the Boltzmann equation allows us to overcome this difficulty.

The following statement establishes existence of a renormalized solution under the assumption of small initial data:

Theorem 1.2 *Let $T>0$ and assume that Assumptions* (A.1) *and* (A.2) *hold. If we denote*:

$$\mathcal{N}_{in}=\int_a^b N_s^{in}\,dx.$$

Then, there exists $\mathcal{N}_0>0$ such that if $\mathcal{N}_{in}\le\mathcal{N}_0$, the system (1.10)–(1.11)–(1.12)–(1.8)–(1.9)–(1.3) *admits a renormalized solution* $(V^\eta,(\epsilon_k^\eta,\chi_k^\eta,f_k^\eta)_{k\ge1})$ *on* $[0,T]$ *which satisfies*

(i) $\forall\lambda>0$, $\Theta_{k,\lambda}^\eta:=(f_k^\eta+\lambda\exp(-\frac{1}{2}(v^2+k^2)))^{1/2}$ *satisfies*

$$\eta\partial_t\Theta_{k,\lambda}^\eta+v\,\partial_x\Theta_{k,\lambda}^\eta-\partial_v(\partial_x\epsilon_k^\eta\,\Theta_{k,\lambda}^\eta)=\frac{Q^\eta(f^\eta)_k}{2\eta\Theta_{k,\lambda}^\eta}+\lambda\partial_x\epsilon_k^\eta\frac{ve^{-\frac{1}{2}(v^2+k^2)}}{2\Theta_{k,\lambda}^\eta}. \tag{1.15}$$

(ii) *We have the local mass conservation*

$$\partial_t N_s^\eta+\partial_x J^\eta=0,\quad \text{where } J^\eta=\frac{1}{\eta}\sum_{k\ge1}\int_{\mathbb{R}}vf_k^\eta\,dv. \tag{1.16}$$

(iii) *The entropy inequality holds*:

$$\forall t\in[0,T],\quad 0\le W^\eta(t)+\frac{\alpha_1}{\eta^2}\int_0^t\mathcal{R}^\eta(s)\,ds\le C_T. \tag{1.17}$$

If the potential is given in L^∞, Poupaud [29] has proved existence of strong solutions of the semiconductors Boltzmann transport equation and their convergence as the mean free path η goes to 0 towards solutions of the drift-diffusion equation. He uses a method based on an asymptotic expansion of the solution f^η in power of η and estimation on the remainder of this expansion. Ben Abdallah and Tayed [6] have extended this method and established the diffusive limit of the Boltzmann-Poisson system in one dimension, since in this case they obtain enough regularity on the potential. However when the dimension is greater than one, Masmoudi and Tayeb [21] need to renormalize the Boltzmann equation and use compactness method to establish the diffusive limit. In this paper we adapt the techniques of Masmoudi and Tayeb [21] to prove the following theorem:

Theorem 1.3 *Let $T>0$ and, for $\eta>0$, $(V^\eta,(f_k^\eta,\epsilon_k^\eta,\chi_k^\eta)_{k\ge1})$ be a renormalized solution of the Boltzmann-Schrödinger-Poisson system as defined in Theorem 1.2 for $\mathcal{N}_{in}\le\mathcal{N}_0$. Then as $\eta\to0$, if $\mathcal{N}_0$ is small enough, this solution converges to a solution $(V,N_s,(\epsilon_k,\chi_k)_{k\ge1})$ of the drift-diffusion-Schrödinger-Poisson* (*DDSP*) *system defined by*

$$\partial_t N_s+\partial_x J=0,\quad J=-\mathbb{D}(\partial_x N_s+N_s\partial_x V_s), \tag{1.18}$$

$$\begin{cases} -\frac{1}{2}\,\partial_{zz}\chi_k + V\chi_k = \epsilon_k\chi_k & (k \geq 1), \\ \chi_k(t,x,\cdot) \in H_0^1(0,1), & \int_0^1 \chi_k\,\chi_\ell\,dz = \delta_{k\ell}, \end{cases} \tag{1.19}$$

$$-\Delta_{x,z}V = N_s\sum_k \frac{e^{-\epsilon_k}}{\sum_\ell e^{-\epsilon_\ell}}\,|\chi_k|^2, \tag{1.20}$$

where the effective potential V_s *is defined by*

$$V_s = -\log\sum_k e^{-\epsilon_k}, \tag{1.21}$$

and $\mathbb{D}$ *is the diffusion coefficient whose expression is given in Corollary 2.2. This system is completed with the initial condition* $N_s(0,x) = N_s^{in}(x)$ *and with the following conservative boundary conditions*:

$$\begin{cases} J(t,a) = J(t,b) = 0, & \frac{dV}{dx}(t,a,z) = \frac{dV}{dx}(t,b,z) = 0, \quad \textit{for } z \in (0,1), \\ V(t,x,0) = V(t,x,1) = 0, & \qquad\qquad\qquad\qquad\qquad\qquad \textit{for } x \in (a,b). \end{cases} \tag{1.22}$$

We have up to an extraction of a subsequence, as $\eta \to 0$,

$$\|f_k^\eta - N_s\mathcal{M}_k\|_{\ell^1(L^1([0,T]\times[a,b]\times\mathbb{R}))} \to 0 \quad \textit{and} \quad \|V^\eta - V\|_{L^2([0,T],H^1(\Omega))} \to 0.$$

We notice the assumption of small initial data in these theorems which has been already set for the study of the Vlasov-Schrödinger-Poisson system in [4]. The existence of solutions for (DDSP) when the x-variable is two dimensional has been established in [5] when the diffusion coefficient $\mathbb{D}$ is assumed to be a constant. In this case we have enough regularity to establish the uniqueness of solutions. But for a non constant diffusion coefficient, the proof of existence is addressed in [34]; however we do not obtain the uniqueness of solutions.

1.4 Strategy of the Proof

As done in [4, 5], the system shall be viewed as a one dimensional Boltzmann equation (1.10) for the distribution function $(f_k^\eta)_{k\geq 1}$ coupled to the quasistatic Schrödinger-Poisson system (1.11)–(1.12) for the potential V^η. The Schrödinger-Poisson system allows us to compute the potential as a function of the distribution function, while the Boltzmann equation gives the value of the distribution function in terms of the electrostatic potential. The arguments used for the proof of Theorem 1.2 are rather standard (see [23] and reference therein). A first step is to truncate and to regularize the Boltzmann-Schrödinger-Poisson system. Thanks to a fixed point argument we can construct strong solutions of the regularized system. Then solutions of the whole system are obtained by a passage to the limit in the regularization using stability result. These steps are explained in Sect. 5.

Theorem 1.3 establishes the diffusive limit of renormalized solutions of Theorem 1.2 as $\eta \to 0$. Regarding the techniques used in the classical Boltzmann-Poisson case [21], the proof of Theorem 1.3 relies strongly on the entropy estimate (1.17) which is established in Sect. 2 and on a rigorous analysis of the Schrödinger-Poisson system. A priori estimates obtained thanks to the entropy allows us to fix the functional framework:

$$(f_k)_{k\geq 1} \in L_t^\infty(L\log L(dx\,dv)), \qquad ((v^2+k^2)f_k)_{k\geq 1} \in L_t^\infty(\ell^1(L^1(dx\,dv))),$$
$$V \in L_t^\infty(H^1(dx\,dv)).$$

As recall in the introduction, a consequence of the Dunford-Pettis and the De La Vallée Poussin Theorem is the relative weak compactness of f^η in $\ell^1(L^1)$. We recall the following averaging lemma whose proof can be found in [21] (see also [8]):

Lemma 1.4 *Assume that h^η is bounded in $L^2((0,T)\times(a,b)\times\mathbb{R})$, that h_0^η and h_1^η are bounded in $L^1((0,T)\times(a,b)\times\mathbb{R})$, and that*

$$\eta\partial_t h^\eta + v\,\partial_x h^\eta = h_0^\eta + \partial_v h_1^\eta.$$

Then for all $\psi \in C_0^\infty(\mathbb{R})$,

$$\lim_{y\to 0}\left(\sup_{\eta<1}\left\|\int_{\mathbb{R}}(h^\eta(t,x+y,v)-h^\eta(t,x,v))\psi(v)\,dv\right\|_{L^1_{t,x}}\right)=0,$$

where h^η is extended by zero for $x\notin[a,b]$.

Thanks to this averaging lemma we will establish in Sect. 4.1 the relative strong compactness of the surface density N_s^η in $\ell^1(L^1)$ as η goes to 0. Then, with the entropy inequality (1.17), we have:

$$\int_0^t \mathcal{R}^\eta(s)\,ds = \frac{1}{2}\sum_k \int_0^t \iint_{(a,b)\times\mathbb{R}}\left(\sqrt{f_k^\eta}-\sqrt{N_s^\eta \mathcal{M}_k^\eta}\right)^2 dx\,dv\,ds \le C_T\eta^2. \tag{1.23}$$

Letting η going to 0 we hope to prove with (1.23) that the distribution function converges to a Maxwellian. But we need to establish the convergence of the eigenenergies ϵ_k^η. Contrary to the Boltzmann-Poisson system [21], the dependency of the potential V^η with respect to the occupation factor ρ^η is not obvious but needs the resolution of the Schrödinger-Poisson system in the functional framework suggested by the a priori estimates.

Therefore a key point is the study of the Schrödinger-Poisson system (1.1)–(1.2), which is the object of Sect. 3. We remark that since we work in one dimension for the transport, we have that $V\in H^1(\Omega)$ implies $\|V\|_{L_z^2(0,1)}\in H^1(a,b)$ which is compactly embedded in $L^\infty(a,b)$. It is proved in the Appendix, where we recall some spectral properties of the Hamiltonian, that it implies a bound on χ_k in $L^\infty(\Omega)$ (see Lemma A.4). Thus the product of ρ_k with $|\chi_k|^2$ in the right hand side of the Poisson equation (1.2) makes sense. Ben Abdallah and Méhats [4] have established existence and uniqueness of solutions of this system (1.1)–(1.2) for an occupation number ρ_k in L^p for $p>1$. The proof is based on an idea of Nier [24, 25] which suggests to minimize the functional

$$J_\rho(V)=\frac{1}{2}\iint_\Omega |\nabla V|^2\,dx\,dz-\sum_{k\ge 1}\int_a^b \rho_k\epsilon_k[V]\,dx.$$

A critical point of this functional is a solution of the Schrödinger-Poisson system. But contrary to [5, 34] where the occupation factors decay with respect to k, this functional is not convex. Thus we do not have uniqueness of the minimum. However we prove in Proposition 3.4 that if $(\rho_k)_{k\ge1}$ and $(\widetilde{\rho}_k)_{k\ge1}$ are in $L^\infty((0,T),\ell^1(L^1(a,b)))$ and if V and $\widetilde{V}$ are corresponding solutions of the Schrödinger-Poisson system (1.1)–(1.2),

$$\|V-\widetilde{V}\|_{L^1([0,T],H^1(\Omega))}\le C_1\|\rho_k-\widetilde{\rho}_k\|_{\ell^1(L^1((0,T)\times(a,b)))}+C_2\mathcal{N}\|V-\widetilde{V}\|_{L^1([0,T],H^1(\Omega))}, \tag{1.24}$$

where $\mathcal{N}=\max\{\|\rho_k\|_{L^\infty((0,T),\ell^1(L^1(a,b)))},\|\widetilde{\rho}_k\|_{L^\infty((0,T),\ell^1(L^1(a,b)))}\}$ and C_1 and C_2 are nonnegative constants depending only on data. We deduce from this inequality that if $\mathcal{N}$ is small enough, the solution of the Schrödinger-Poisson system (1.1)–(1.2) is unique. It explains why Theorems 1.2 and 1.3 are proved only under the assumption of small initial data.

Yet we can prove that the strong compactness of N_s^η in L^1 implies the strong compactness of V^η in $L^1((0,T),H^1(\Omega))$. From spectral properties of the Hamiltonian it implies that $\epsilon_k[V^\eta] \to \epsilon_k[V]$ as η goes to 0. From (1.23) we deduce that $f^\eta \to N_s\mathcal{M}$ in $\ell^1(L^1(dt\,dx\,dv))$. It remains to show that the limit function N_s is a solution of the drift-diffusion equation (1.18). Passing to the limit in the local mass conservation, it suffices to study the limit of the current J^η which is done in Sect. 4.2.

The outline of the paper is as follows. In the second section, after briefly recalling basic properties of the collision operator, we establish the a priori estimates, which are the natural estimates for our system. In the third section, we analyze the Schrödinger-Poisson system under physical assumptions given by the a priori estimates. Section 4 is devoted to the proof of Theorem 1.3 assuming that we have constructed a renormalized solution of the Boltzmann-Schrödinger-Poisson system. In Sect. 5, the proof of Theorem 1.2 is considered: we give the regularization and explain the passing to the limit in the regularized system. The Appendix is devoted to some useful properties on the spectrum of the Schrödinger operator.

2 A Priori Estimate

2.1 Properties of the Collision Operator

This section is devoted to the study of the collision operator defined by (1.5). The collision operator Q operates on the v variable only, then we omit in this section the spatial and time dependency, since these variables are only parameters. We assume that the sequence $(\epsilon_k)_{k\ge 1}$ is given and we define $\mathcal{M}_k(v) = \frac{1}{2\pi\mathcal{Z}}\exp(-\frac{1}{2}v^2 - \epsilon_k)$ for $\mathcal{Z} = \sum_{k\ge 1} e^{-\epsilon_k}$. We introduce the space:

$$L^2_{\mathcal{M}} = \left\{ (f_k)_{k\in\mathbb{N}^*} \text{ s.t. } \sum_k \int_{\mathbb{R}} f(v)^2/\mathcal{M}_k(v)\,dv < +\infty \right\}, \tag{2.1}$$

with the associated inner product:

$$\langle f, g\rangle_{\mathcal{M}} = \sum_k \int_{\mathbb{R}} \frac{f_k g_k}{\mathcal{M}_k}\,dv.$$

Then we summarize the main properties of this collision operator in the following proposition.

Proposition 2.1 *Let Q be defined by* (1.5) *with a cross section α symmetric and bounded from above and below i.e. satisfying* (A.1). *Then we get*:

(i) $\sum_k \int Q(f)_k(v)\,dv = 0$.
(ii) *Q is a linear, bounded, selfadjoint and negative operator on $L^2_{\mathcal{M}}$.*
(iii) *The nullspace*: $\mathrm{Ker}\, Q = \{f \in L^2_{\mathcal{M}}$ *s.t.* $\exists N_s \in \mathbb{R}$ *with* $f_k = N_s\mathcal{M}_k,\ \forall k \ge 1\}$.
(iv) *The equation $Q(f) = g$ admits a solution $f \in L^2_{\mathcal{M}}$ iff*

$$\sum_k \int_{\mathbb{R}} g_k(v)\,dv = 0,$$

and this solution is unique if we impose the same relation on f.

Proof The first point is trivial. Using the symmetry of the cross section, we get the crucial identity:

$$2\langle Q(f), g\rangle_M = -\sum_{k,k'} \iint \alpha_{k,k'} \mathcal{M}_k(v)\mathcal{M}_{k'}(v') \times \left(\frac{f_{k'}(v')}{\mathcal{M}_{k'}(v')} - \frac{f_k(v)}{\mathcal{M}_k(v)}\right)\left(\frac{g_{k'}(v')}{\mathcal{M}_{k'}(v')} - \frac{g_k(v)}{\mathcal{M}_k(v)}\right) dv\, dv'.$$

Then (ii) and (iii) are easy consequences from this identity. It follows,

$$(\mathrm{Ker}\, Q)^\perp = \left\{ f \in L^2_{\mathcal{M}} \text{ s.t. } \sum_k \int f_k(v)\, dv = 0 \right\}.$$

Since Q is obviously a closed operator in $L^2_{\mathcal{M}}$ the equation $Q(f) = g$ admits a solution iff $g \in (\mathrm{Ker}\, Q)^\perp$. This solution is unique in $(\mathrm{Ker}\, Q)^\perp$. □

Corollary 2.2 *There exists $\Theta \in L^2_{\mathcal{M}}$ such that for all $k \geq 1$,*

$$Q(\Theta)_k = -v\mathcal{M}_k \quad \textit{and} \quad \sum_k \int_{\mathbb{R}} \Theta_k\, dv = 0.$$

Then we can define the diffusion coefficient as

$$\mathbb{D} = \sum_k \int_{\mathbb{R}} \Theta_k \otimes v\, dv. \tag{2.2}$$

Remark 2.3 We recognize in formula (2.2) the classical expression for the diffusion coefficient in all the problem of approximation of transport process by diffusion. This formula, known as the Kubo's formula, is still valid in higher dimensions and under Assumption (A.1) on the cross-section it defines a positive definite matrix [17].

2.2 A Priori Estimate

A key argument in our study is to obtain uniform estimates on the unknows of the system. We use the entropy defined in (1.13). All along the paper, we will use the following functional space:

$$L^p_x L^q_z(\Omega) = \left\{ u \in L^1_{loc}(\Omega) \text{ such that } \|u\|_{L^p_x L^q_z(\Omega)} = \left(\int_a^b \|u(x,\cdot)\|^p_{L^q_z(0,1)}\, dx\right)^{1/p} < +\infty \right\}.$$

We recall (see Lemma 2.2 of [4]).

Lemma 2.4 *Let $\Omega = (a,b) \times (0,1) \subset \mathbb{R}^2$. Then the space $H^1(\Omega)$ is continuously imbedded in $L^\infty_x L^2_z(\Omega)$.*

We notice that this embedding does not hold if $\Omega = \omega \times (0,1)$ for ω a bounded domain of $\mathbb{R}^2$, i.e. if the transport is assumed to take place in a bounded domain of $\mathbb{R}^2$.

Proposition 2.5 *Let $T > 0$ and let $(V^\eta, (f_k^\eta, \epsilon_k^\eta, \chi_k^\eta)_{k\geq 1})$ be a renormalized solution on the interval $[0, T]$ of the Boltzmann-Schrödinger-Poisson system (1.10)–(1.1)–(1.2) with boundary conditions (1.9)–(1.8). We assume that* (A.1) *and* (A.2) *hold and that*

$$((1 + v^2 + \epsilon_k^\eta + \log f_k^\eta) f_k^\eta)_{k\geq 1} \in L^\infty([0, T], \ell^1(L^1((a, b) \times \mathbb{R})))$$

and

$$V^\eta \in L^\infty([0, T], H^1(\Omega)).$$

Then, there exists a nonnegative constant C depending only on initial data such that,

$$\forall t \in [0, T], \quad 0 \leq W^\eta(t) + \frac{\alpha_1}{\eta^2} \int_0^t \mathcal{R}^\eta(s)\, ds \leq C, \tag{2.3}$$

where the entropy W^η is defined in (1.13) *and the dissipation rate $\mathcal{R}^\eta$ is given in* (1.14). *Moreover,*

$$\forall t \in [0, T], \quad \int_a^b N_s^\eta(t, x) = \mathcal{N}_{in} = \int_a^b N_s^{in}(x)\, dx. \tag{2.4}$$

Proof This result is proved in the case of smooth solutions for which all calculations are justified. In a general case, we regularize the system to have smooth solutions and pass to the limit in the estimate obtained for these smooth solutions. These steps are explained in Sect. 5.2.

It is readily seen that with our assumption on the initial condition (A.2), the initial entropy is bounded and that with our boundary conditions, the system conserves the mass which implies (2.4). Multiplying (1.10) by $(1 + \log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta)$, integrating on $(a, b) \times \mathbb{R}$ and summing over k, we get

$$\begin{aligned}
&\sum_k \iint \partial_t f_k^\eta \left(\log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta + 1\right) dx\, dv \\
&\quad = \frac{d}{dt} \sum_k \iint f_k^\eta \left(\log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta\right) dx\, dv - \sum_k \iint f_k^\eta \partial_t \epsilon_k^\eta\, dx\, dv.
\end{aligned}$$

Moreover, using the notation $\langle f \rangle = \int_0^1 f(z)\, dz$, we have $\partial_t \epsilon_k^\eta = \langle |\chi_k^\eta|^2 \partial_t V^\eta \rangle$ (see Lemma A.2 in the Appendix). Thus we obtain:

$$\begin{aligned}
&\sum_k \iint f_k^\eta \partial_t \epsilon_k^\eta\, dx\, dv \\
&\quad = \sum_k \iiint f_k^\eta |\chi_k^\eta|^2 \partial_t V^\eta\, dx\, dv\, dz \\
&\quad = \frac{d}{dt} \sum_k \iint f_k^\eta \langle |\chi_k^\eta|^2 V^\eta \rangle\, dx\, dv - \frac{1}{2} \frac{d}{dt} \iint |\nabla_{x,z} V^\eta|^2\, dx\, dz,
\end{aligned}$$

where we use the Poisson equation (1.2). Therefore,

$$\sum_k \iint \partial_t f_k^\eta \left(\log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta + 1\right) dx\, dv$$
$$= \frac{1}{2}\frac{d}{dt}\iint |\nabla_{x,z} V^\eta|^2\, dx\, dz$$
$$+ \frac{d}{dt}\sum_k \iint f_k^\eta \left(\log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta - \langle |\chi_k^\eta|^2 V^\eta\rangle\right) dx\, dv. \tag{2.5}$$

And from the Schrödinger equation (1.1) we have:

$$\frac{1}{2}\langle |\partial_z \chi_k^\eta|^2\rangle + \langle |\chi_k^\eta|^2 V^\eta\rangle = \epsilon_k^\eta.$$

With our boundary condition (1.8) we have after an integration by parts

$$\sum_k \iint \left(v\cdot\partial_x f_k^\eta + \partial_x \epsilon_k^\eta \cdot \partial_v f_k^\eta\right)\left(\log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta + 1\right) dx\, dv$$
$$= \left[\sum_k \int_{\mathbb{R}} v f_k^\eta \left(\log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta\right) dv\right]_a^b = 0. \tag{2.6}$$

Finally, with (1.5) and since $\sum_k \int Q^\eta(f^\eta)_k\, dv = 0$,

$$\sum_k \int Q^\eta(f^\eta)_k \left(\log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta + 1\right) dv$$
$$= \frac{1}{2}\sum_{k,k'} \iint \alpha_{k,k'}(\mathcal{M}_k^\eta(v) f_{k'}^\eta(v') - \mathcal{M}_{k'}^\eta(v') f_k^\eta(v)) \log\left[\left(\frac{f_k^\eta(v)}{\mathcal{M}_k^\eta(v)}\right)\left(\frac{\mathcal{M}_{k'}^\eta(v')}{f_{k'}^\eta(v')}\right)\right] dv\, dv'.$$

Using the relation $(a_1 - a_2)\log(a_1/a_2) \geq (\sqrt{a_1} - \sqrt{a_2})^2$, for all positive a and b, and the Jensen inequality, we obtain:

$$\sum_k \iint Q^\eta(f^\eta)_k \left(\log f_k^\eta + \frac{|v|^2}{2} + \epsilon_k^\eta + 1\right) dv\, dx \leq -\alpha_1 \mathcal{R}^\eta(t). \tag{2.7}$$

Finally, (2.5), (2.6) and (2.7) lead to:

$$\frac{d}{dt}\sum_k \iint f_k^\eta \left(\log f_k^\eta + \frac{|v|^2}{2} + \frac{1}{2}\langle |\partial_z \chi_k^\eta|^2\rangle\right) dx\, dv$$
$$+ \frac{1}{2}\frac{d}{dt}\iint |\nabla_{x,z} V^\eta|^2\, dx\, dz + \frac{\alpha_1}{\eta^2}\mathcal{R}^\eta(t) \leq 0. \tag{2.8}$$

From (2.8) we have after an integration on $[0, T]$,

$$\sum_k \iint f_k^\eta \left(\log f_k^\eta + \frac{|v|^2}{2} + \frac{k^2}{2} - 1\right) dx\, dv + \frac{1}{2}\iint |\nabla_{x,z} V^\eta|^2\, dx\, dz$$
$$+ \frac{\alpha_1}{\eta^2}\int_0^T \mathcal{R}^\eta(t)\, dt \leq C_1 + \sum_k \int_a^b \rho_k^\eta \left(\frac{k^2}{2} - \frac{1}{2}\langle |\partial_z \chi_k^\eta|^2\rangle\right) dx. \tag{2.9}$$

Moreover, since the potential V^η is nonnegative, we have with the Hölder inequality

$$\frac{1}{2}\langle|\partial_z\chi_k^\eta|^2\rangle = \epsilon_k^\eta - \langle|\chi_k^\eta|^2 V^\eta\rangle \ge \epsilon_k[0] - \|\chi_k^\eta\|^2_{L^4_z(0,1)}\|V^\eta\|_{L^2_z(0,1)}.$$

An interpolation and Lemma A.4 imply the existence of a nonnegative constant C_2 such that

$$\|\chi_k^\eta\|^2_{L^4_z(0,1)} \le C\|\chi_k^\eta\|_{L^2_z(0,1)}\|\chi_k^\eta\|_{L^\infty_z(0,1)} \le C_2(1+\|V^\eta\|^{1/2}_{L^2_z(0,1)}).$$

Since $\epsilon_k[0] = \frac{1}{2}\pi^2k^2$, we deduce that

$$\frac{k^2}{2} - \frac{1}{2}\langle|\partial_z\chi_k^\eta|^2\rangle \le \frac{1}{2}k^2 - \frac{1}{2}\pi^2k^2 + C_2(1+\|V^\eta\|^{1/2}_{L^2_z(0,1)}) \le C_2(1+\|V^\eta\|^{1/2}_{L^2_z(0,1)}). \quad (2.10)$$

By the Sobolev embedding $H^1(\Omega)\hookrightarrow L^\infty_x L^2_z(\Omega)$, we have

$$\begin{aligned}\sum_k\int_a^b \rho_k^\eta\left(\frac{k^2}{2} - \frac{1}{2}\langle|\partial_z\chi_k^\eta|^2\rangle\right)dx &\le C_3\|\rho_k^\eta\|_{\ell^1(L^1(a,b))}(1+\|V^\eta\|^{1/2}_{H^1(\Omega)})\\ &= C_3\mathcal{N}_{in}(1+\|V^\eta\|^{1/2}_{H^1(\Omega)}). \end{aligned} \quad (2.11)$$

This last inequality in (2.9) provides

$$\iint |\nabla_{x,z}V^\eta|^2\,dx\,dz \le C_4 + C_5\|V^\eta\|^{1/2}_{H^1(\Omega)}.$$

Thus using the Poincaré inequality, we deduce that $\|V^\eta\|_{H^1(\Omega)}$ is bounded. Then (2.9) and (2.11) provide the desired estimate. □

Corollary 2.6 *Let $T>0$ and $(f_k^\eta)_{k\ge1}$ such as in Proposition 2.5, there exists a constant $C_T>0$ such that*:

$$\forall t\in[0,T],\quad \sum_k\iint_{(a,b)\times\mathbb{R}} f_k^\eta(|\log f_k^\eta| + |v|^2 + k^2 + 1)\,dx\,dv \le C_T,$$

$$\int_a^b (N_s^\eta\log N_s^\eta - N_s^\eta + 1)\,dx \le C_T,\qquad \int_0^T\int_a^b J^\eta(t,x)\,dx\,dt \le C_T.$$

Proof The second estimate results from the Jensen inequality. The first follows from the remark $y|\log y| \le y\log y + 2/e$ for all $y>0$. Since the function $v\mapsto v\mathcal{M}_k^\eta$ is odd, we have

$$J^\eta = \frac{1}{\eta}\sum_k\int_{\mathbb{R}} v\left(\sqrt{f_k^\eta} + \sqrt{N_s^\eta\mathcal{M}_k^\eta}\right)\left(\sqrt{f_k^\eta} - \sqrt{N_s^\eta\mathcal{M}_k^\eta}\right)dv.$$

Using the Cauchy-Schwarz inequality, we deduce that

$$\int_a^b J^\eta(t,x)\,dx \le 2\left(\sum_k\iint v^2(f_k^\eta + N_s^\eta\mathcal{M}_k^\eta)\,dx\,dv\right)^{1/2}(\mathcal{R}^\eta(t))^{1/2}.$$

We conclude by using (2.3). □

Remark 2.7 It could seem more "*natural*" to consider the relative entropy with respect to the physical equilibrium $\mathcal{M}$ rather than W^η in (1.13). However it this case the time derivative of the entropy with respect to time will involve terms $\partial_t \epsilon_k$ for which we have no estimate. This is the reason why we choose the time-independent quantity M in (1.13).

3 The Schrödinger-Poisson System

This section is devoted to the study of the "quasi-static" Schrödinger-Poisson system defined by:

$$\begin{cases} -\frac{1}{2}\partial_z^2 \chi_k + V\chi_k = \epsilon_k \chi_k & (k \geq 1), \\ \chi_k(t,x,\cdot) \in H_0^1(0,1), & \int_0^1 \chi_k\, \chi_\ell \, dz = \delta_{k\ell}, \end{cases} \tag{3.1}$$

$$-\Delta_{x,z} V = \sum_k \rho_k |\chi_k|^2, \tag{3.2}$$

where we consider that $\rho = (\rho_k)_{k\geq 1}$ is given in $L^\infty((0,T), \ell^1(L^1(a,b)))$ and satisfies:

(H1) $\forall k \geq 1$, $\rho_k \geq 0$ and there exists a nonnegative constant C_T such that

$$\forall t \in [0,T], \quad \sum_k \int_a^b \rho_k (1+k^2)\, dx \leq C_T. \tag{3.3}$$

We denote $N_s = \sum_k \rho_k$. The system is completed by the boundary conditions (1.3). In the sequel we will use the functional space $H_{01}^1 = \{V \in H^1(\Omega) : V(x,0) = V(x,1) = 0\}$.

Proposition 3.1 (Existence and uniqueness) *Let us suppose that $\rho = (\rho_k)_{k\geq 1}$ is given in $L^\infty((0,T), \ell^1(L^1(a,b)))$ and satisfies* H1. *Then the Schrödinger-Poisson system* (3.1)–(3.2) *admits a solution in H_{01}^1.*

Moreover, denoting $\mathcal{N} = \|N_s\|_{L^\infty((0,T),L^1(a,b))}$ if $\mathcal{N}$ is small enough, this solution $(V, (\epsilon_k, \chi_k)_{k\geq 1})$ is unique.

This result is obtained thanks to an idea of Nier [24] which has been developed in [4]. The principle is based on the fact that a weak solution of (3.1)–(3.2) is a critical point of a certain functional. Namely, we consider the functional defined on H_{01}^1 by

$$J_\rho(V) = \frac{1}{2}\iint_\Omega |\nabla V|^2 \, dx\, dz - \sum_{k\geq 1} \int_a^b \rho_k \epsilon_k[V]\, dx = J_0(V) + J_1(V,\rho). \tag{3.4}$$

It is proved in Lemma 3.2 that this functional admits a minimizer and that this minimizer is a weak solution of (3.1)–(3.2). Because of the non-convexity of J_ρ, its minimizers are not unique. Hence the uniqueness is obtained in Lemma 3.3 only under the assumption of smallness for $\mathcal{N}$.

Lemma 3.2 *Assume that $(\rho_k)_{k\geq 1} \in L^\infty((0,T), \ell^1(L^1(a,b)))$ and satisfy* H1. *Then the functional J_ρ defined in* (3.4) *is continuous, locally Lipschitz and weakly lower semicontinuous on H_{01}^1. It is coercive: there exist nonnegative constants C_1, C_2 and C_3 such that for all $t \in (0,T)$,*

$$J_\rho(V) \geq C_1 \|V\|_{H^1(\Omega)}^2 - C_2 \|V\|_{H^1(\Omega)}^{3/2} - C_3. \tag{3.5}$$

Thus the system (3.1)–(3.2) *admits a solution $(V, (\epsilon_k, \chi_k)_{k\geq 1})$ with $V \in L^\infty((0,T), H_{01}^1)$.*

Proof The functional J_0 is clearly continuous and strongly convex on H^1_{01}. For the functional J_1, we use the properties of $\epsilon_k[V]$ summarized in (A.8) to prove

$$
\begin{aligned}
|J_1(V,\rho) - J_1(\widetilde{V},\rho)| &\le \sum_{k\ge 1}\int_a^b \rho_k|\epsilon_k[V]-\epsilon_k[\widetilde{V}]|\,dx \\
&\le C_1\sum_{k\ge 1}\int_a^b \rho_k(1+\|V\|^{1/2}_{L^2_z(0,1)}+\|\widetilde{V}\|^{1/2}_{L^2_z(0,1)})\|V-\widetilde{V}\|_{L^2_z(0,1)}\,dx.
\end{aligned}
\tag{3.6}
$$

If we use the Sobolev embedding stated in Lemma 2.4, we obtain

$$
|J_1(V,\rho)-J_1(\widetilde{V},\rho)| \le C_2(1+\|V\|^{1/2}_{H^1(\Omega)}+\|\widetilde{V}\|^{1/2}_{H^1(\Omega)})\|N_s\|_{L^1(a,b)}\|V-\widetilde{V}\|_{H^1(\Omega)}. \tag{3.7}
$$

Hence $J_1(\cdot,\rho)$ is Lipschitz and weakly continuous on H^1_{01}. Now if we take $\widetilde{V}=0$ in (3.7), from H1, we have that $0\ge J_1(0,\rho)\ge -C_T$. Thus,

$$
J_\rho(V) \ge \frac{1}{2}\|\nabla V\|^2_{L^2(\Omega)} - C_3(1+\|V\|^{1/2}_{H^1(\Omega)})\|V\|_{H^1(\Omega)} - C_4.
$$

We apply the Poincaré inequality in H^1_{01} to find (3.5). Hence the functional J_ρ admits a minimizer in H^1_{01}. Moreover, from Lemma A.2, it is clear that J_ρ is Gâteaux differentiable on H^1_{01} and the differential of J_ρ in the direction $W\in H^1(\Omega)$ is:

$$
d_V J_\rho(V)\cdot W = \iint_\Omega \nabla V\cdot\nabla W\,dx\,dz - \sum_k\int_a^b \rho_k\langle|\chi_k[V]|^2 W\rangle\,dx.
$$

Thus each minimizer of the functional J_ρ is a weak solution of the Schrödinger-Poisson system (3.1)–(3.2). ☐

Lemma 3.3 *Let* $(\rho_k)_{k\ge1}$ *given in* $L^\infty((0,T),\ell^1(L^1(a,b)))$ *and satisfying* H1. *Then, for* $\mathcal{N}:=\|N_s\|_{L^\infty((0,T),L^1(a,b))}$ *small enough, the corresponding solution* $(V,(\epsilon_k[V],\chi_k[V])_{k\ge1})$ *of the Schrödinger-Poisson system* (3.1)–(3.2) *is unique.*

Proof Let $(\rho_k)_{k\ge1}$ be in $L^\infty((0,T),\ell^1(L^1(a,b)))$ satisfying H1. We assume that we can find two solutions of the Schrödinger–Poisson system denoted V and $\widetilde{V}$. Multiplying the Poisson equation (3.2) by $(\widetilde{V}-V)$ and integrating provides:

$$
\iint_\Omega |\nabla(\widetilde{V}-V)|^2\,dx\,dz = \sum_k\int_a^b \rho_k\langle(|\chi_k[\widetilde{V}]|^2-|\chi_k[V]|^2)(\widetilde{V}-V)\rangle\,dx. \tag{3.8}
$$

From (A.3), we deduce that we have

$$
\iint_\Omega |\nabla(\widetilde{V}-V)|^2\,dx\,dz \le C_1\int_a^b N_s\,e^{C_2(\|V\|_{L^2_z(0,1)}+\|\widetilde{V}\|_{L^2_z(0,1)})}\|V-\widetilde{V}\|^2_{L^2_z(0,1)}\,dx.
$$

Then the Sobolev embedding $H^1(\Omega)\hookrightarrow L^\infty_x L^2_z(\Omega)$ and the Poincaré inequality lead to

$$
\|V-\widetilde{V}\|^2_{H^1(\Omega)} \le C_2 e^{C_4(\|V\|_{H^1(\Omega)}+\|\widetilde{V}\|_{H^1(\Omega)})}\|N_s\|_{L^1(a,b)}\|V-\widetilde{V}\|^2_{H^1(\Omega)}. \tag{3.9}
$$

From Lemma 3.2 we know that V and $\widetilde{V}$ are bounded in $H^1(\Omega)$. Thus, there exists a nonnegative constant C_3 such that

$$\|V-\widetilde{V}\|^2_{H^1(\Omega)} \le C_3\mathcal{N}\|V-\widetilde{V}\|^2_{H^1(\Omega)}. \tag{3.10}$$

Thus it suffices to chose $\mathcal{N}$ small enough such that $C_3\mathcal{N} \le 1/2$ to prove that $V = \widetilde{V}$ on $[0, T] \times \Omega$. □

Proposition 3.4 (Continuity) *Let $(\rho_k)_{k\ge1}$ and $(\widetilde{\rho}_k)_{k\ge1}$ in $L^\infty((0,T), \ell^1(L^1(a,b)))$ and satisfying* H1. *We denote by $\mathcal{N} := \|N_s\|_{L^\infty((0,T),L^1(a,b))}$, $\widetilde{\mathcal{N}} := \|\widetilde{N}_s\|_{L^\infty((0,T),L^1(a,b))}$, V and $\widetilde{V}$ the corresponding solutions of the Schrödinger-Poisson system* (3.1)–(3.2). *Then there exists $\mathcal{N}_0$ such that if* $\max(\mathcal{N}, \widetilde{\mathcal{N}}) \le \mathcal{N}_0$, *then for all $p \ge 1$*

$$\|V-\widetilde{V}\|_{L^p([0,T],H^1(\Omega))} \le C_T\|\rho_k-\widetilde{\rho}_k\|_{L^p([0,T],\ell^1(L^1(a,b)))},$$

where C_T is a nonnegative constant depending only on T.

Proof Let $(\rho_k)_{k\ge1}$ and $(\widetilde{\rho}_k)_{k\ge1}$ be two sequences in $L^\infty((0,T), \ell^1(L^1(a,b)))$ satisfying H1. Multiplying the Poisson equation (3.2) by $(V-\widetilde{V})$ and integrating provides:

$$\begin{aligned}\int\!\!\int_\Omega |\nabla(V-\widetilde{V})|^2\,dx\,dz &= \sum_k \int\!\!\int_\Omega (\rho_k-\widetilde{\rho}_k)|\chi_k[V]|^2(V-\widetilde{V})\,dx\,dz \\ &\quad + \sum_k \int_a^b \widetilde{\rho}_k\langle(|\chi_k[V]|^2-|\chi_k[\widetilde{V}]|^2)(V-\widetilde{V})\rangle\,dx. \end{aligned} \tag{3.11}$$

We treat the second term as in the proof of Lemma 3.3 and obtain:

$$\sum_k \int_a^b \widetilde{\rho}_k\langle(|\chi_k[V]|^2-|\chi_k[\widetilde{V}]|^2)(V-\widetilde{V})\rangle\,dx \le C_1\widetilde{\mathcal{N}}\|V-\widetilde{V}\|^2_{H^1(\Omega)}, \tag{3.12}$$

where C_1 is a nonnegative constant. For the first term, we have with Lemma A.1

$$\sum_k \int\!\!\int_\Omega (\rho_k-\widetilde{\rho}_k)|\chi_k[V]|^2(V-\widetilde{V})\,dx\,dz \le C_2\int_a^b \sum_{k\ge1}|\rho_k-\widetilde{\rho}_k|e^{C_3\|V\|_{L^2_z}}\|V-\widetilde{V}\|_{L^2_z}\,dx.$$

And by the Sobolev embedding $H^1(\Omega) \hookrightarrow L^\infty_x L^2_z(\Omega)$ and the bound of V and $\widetilde{V}$ in $H^1(\Omega)$, we have

$$\sum_{k\ge1}\int_a^b (\rho_k-\widetilde{\rho}_k)|\chi_k[V]|^2(V-\widetilde{V})\,dx\,dz \le C_4\|\rho_k-\widetilde{\rho}_k\|_{\ell^1(L^1(a,b))}\|V-\widetilde{V}\|_{H^1(\Omega)}. \tag{3.13}$$

Therefore if we inject (3.12) and (3.13) in (3.11), we obtain thanks to the Poincaré inequality:

$$\|V-\widetilde{V}\|_{H^1(\Omega)} \le C_5\mathcal{N}\|V-\widetilde{V}\|_{H^1(\Omega)} + C_6\|\rho_k-\widetilde{\rho}_k\|_{\ell^1(L^1(a,b))}.$$

The result follows straightforwardly after an integration in time for $\mathcal{N}_0$ small enough. □

4 Diffusive Limit

In this section we prove Theorem 1.3 assuming that we have constructed a renormalized solution $((f_k^\eta)_{k\geq 1}, V^\eta)$ of the Boltzmann-Schrödinger-Poisson system (1.10)–(1.12) such as in Theorem 1.2.

Adapting the arguments in [21], we prove in a first subsection the convergence up to an extraction of the solution $((f_k^\eta)_{k\geq 1}, V^\eta)$ as η goes to 0. In a second subsection, we show that the limit is a solution of the (DDSP) system.

4.1 Convergence of the Renormalized Solutions

Let f^η be a renormalized solution of the Boltzmann equation. The a priori estimates of Corollary 2.6 imply that f^η is weakly relatively compact in $\ell^1(L^1([0,T]\times(a,b)\times\mathbb{R}))$. The two following lemmata show that we can apply the averaging Lemma 1.4 and that it implies the strong convergence of N_s^η. The convergence of (f^η, V^η) is then proved in Proposition 4.3 using the smallness assumption on initial data.

Let us denote, for $\delta > 0$ fixed, β_δ an approximation of the identity, namely $\beta_\delta(s) = \frac{1}{\delta}\beta(\delta s)$. We choose β and C^∞ function satisfying $\beta(s) = s$ for $s \leq 1$, $0 \leq \beta'(s) \leq 1$ for all s and $\beta(s) = 2$ for $s \geq 3$.

Lemma 4.1 *Let f^η be a renormalized solution of the Boltzmann equation such as in Theorem 1.2. Then $\frac{Q^\eta(f^\eta)}{\eta}$ is weakly relatively compact in $\ell^1(L^1((0,T)\times(a,b)\times\mathbb{R}))$.*

Proof We define

$$r_k^\eta = \frac{\sqrt{f_k^\eta} - \sqrt{N_s^\eta \mathcal{M}_k^\eta}}{\eta\sqrt{\mathcal{M}_k^\eta}}. \tag{4.1}$$

Thanks to the dissipation rate control (1.17), we have

$$\sum_k \int_0^T \iint |r_k^\eta|^2 \mathcal{M}_k^\eta \, dx\, dv\, dt \leq C. \tag{4.2}$$

Using r^η we can rewrite

$$f_k^\eta = N_s^\eta \mathcal{M}_k^\eta + 2\eta\sqrt{N_s^\eta}\mathcal{M}_k^\eta r_k^\eta + \eta^2 (r_k^\eta)^2 \mathcal{M}_k^\eta.$$

The result is then obtained thanks to a straightforward adaptation of the proof of Proposition 3.3 in [21]. □

Lemma 4.2 *Let $N_s^\eta = \sum_k \int f_k^\eta \, dv$ with f^η such as in Theorem 1.2. Then N_s^η is relatively compact in $L^1((0,T)\times(a,b))$.*

Proof We can rewrite the renormalized Boltzmann equation:

$$\eta\partial_t \beta_\delta(f_k^\eta) + v\cdot\partial_x \beta_\delta(f_k^\eta) = h_k^\eta + \partial_v g_k^\eta,$$

where

$$h_k^\eta = \frac{1}{\eta} Q^\eta(f^\eta)_k \beta_\delta'(f_k^\eta) \quad \text{and} \quad g_k^\eta = \partial_x \epsilon_k^\eta \beta_\delta(f_k^\eta).$$

With our choice of β_δ, we have $\beta_\delta(f_k^\eta) \le 2/\delta$ and $\beta_\delta(f_k^\eta) \le f_k^\eta$ then $\beta_\delta(f^\eta) \in \ell^\infty(L^\infty_{t,x,v}) \cap \ell^1(L^1_{t,x,v})$. It yields that $\beta_\delta(f^\eta) \in \ell^2(L^2_{t,x,v})$. Since we have $0 \le \beta'_\delta(f_k^\eta) \le 1$ and $\frac{1}{\eta}Q^\eta(f^\eta)$ weakly relatively compact in $\ell^1(L^1_{t,x,v})$, we deduce that h_k^η is weakly relatively compact in $\ell^1(L^1_{t,x,v})$. The spectral properties of the Hamiltonian (see Lemma A.2) imply $\partial_x \epsilon_k = \langle |\chi_k|^2 \partial_x V\rangle$. From Lemma A.4 and the Cauchy-Schwarz inequality, we deduce

$$\sum_{k\ge 1}\int |\partial_x \epsilon_k^\eta \beta_\delta(f_k^\eta)|\,dx\,dv \le \frac{C}{\sqrt{\delta}}(1+\|V^\eta\|_{H^1(\Omega)})\|V^\eta\|_{H^1(\Omega)}\left(\sum_k \int |\beta_\delta(f_k^\eta)|\,dx\,dv\right)^{1/2}.$$

The bound of V^η in $H^1(\Omega)$ and of f^η in $\ell^1(L^1_{t,x,v})$ implies that g_k^η is bounded in $\ell^1(L^1_{t,x,v})$.

Thus we can apply the averaging Lemma 1.4. We have that for all $\psi_k \in \mathcal{D}(\mathbb{R})$ with $(\psi_k)_{k\ge 1}$ all null except for a finite number of them,

$$\lim_{y\ge 0}\left(\sup_{\eta\le 1}\left\|\sum_{k\ge 1}\int_{\mathbb{R}} (\beta_\delta(f_k^\eta)(t,x+y,v) - \beta_\delta(f_k^\eta)(t,x,v))\psi_k(v)\,dv\right\|_{L^1_{t,x}}\right) = 0. \tag{4.3}$$

Next, using the fact that $((1+v^2+k^2)\beta_\delta(f_k^\eta))_\eta$ is bounded in $L^\infty(0,T;\ell^1(L^1_{x,v}))$, we deduce from standard argument that we can take $\psi_k(v)$ to be constant equal to 1 in (4.3). Moreover the definition of β_δ and the equi-integrability of f_k^η implies

$$\sup_{\eta\le 1}\|\beta_\delta(f^\eta) - f^\eta\|_{\ell^1(L^1_{t,x,v})} \to 0 \quad \text{as } \delta \to 0. \tag{4.4}$$

Let $\varepsilon > 0$, we have for all $1 \ge \eta > 0$

$$\begin{aligned}
&\int |N_s^\eta(t,x+y) - N_s^\eta(t,x)|\,dt\,dx \\
&\quad \le \sum_k \int |f_k^\eta(t,x+y,v) - \beta_\delta(f_k^\eta)(t,x+y,v)|\,dt\,dx\,dv \\
&\qquad + \sum_k \int |\beta_\delta(f_k^\eta) - f_k^\eta|\,dt\,dx\,dv \\
&\qquad + \int \left|\sum_k \int_{\mathbb{R}} \beta_\delta(f_k^\eta)(t,x+y,v)\,dv - \sum_k \int_{\mathbb{R}} \beta_\delta(f_k^\eta)(t,x,v)\,dv\right| dt\,dx.
\end{aligned}$$

We fix δ such that the first and the second term of the right hand side is $< \varepsilon/3$. For such a $\delta > 0$, we use (4.3) to bound the third term by $\varepsilon/3$ for y small enough. Then

$$\|N_s^\eta(t,x+y) - N_s^\eta(t,x)\|_{L^1_{t,x}} \to 0 \quad \text{when } y \to 0 \text{ uniformly in } \eta.$$

Therefore the sequence $(N_s^\eta(t,\cdot))_\eta$ is relatively compact in L^1_x for all $t \in [0,T]$. From the local mass conservation (1.16), we obtain that $\partial_t N_s^\eta = -\partial_x J^\eta$, which is bounded in $L^1(0,T;W^{-1,1}(a,b))$ thanks to Corollary 2.6. We deduce the relative strong compactness of $(N_s^\eta)_\eta$ in $L^1_{t,x}$. Therefore we can extract a subsequence such that $N_s^\eta \to N_s$ in $L^1((0,T)\times(a,b))$ and a.e. By uniqueness of the weak limit, there exists $\rho \in \ell^1(L^1((0,T)\times(a,b)))$ such that $N_s = \sum_k \rho_k$ and $\rho_k^\eta \rightharpoonup \rho_k$ weakly in $\ell^1(L^1_{t,x})$. □

Proposition 4.3 *Let (f^η, V^η) be a renormalized solution of the coupled Boltzmann-Schrödinger-Poisson system which satisfies* (i), (ii) *and* (iii) *of Theorem* 1.2. *There exist*

V in $L^\infty((0,T),H^1(\Omega))$ and N_s in $L^\infty((0,T),L^1(a,b))$ such that if $\mathcal{N}_{in}$ is small enough, then up to an extraction we have

$$V^\eta \to V \quad \textit{in } L^2((0,T),H^1(\Omega)) \quad \textit{and}$$
$$f^\eta \to N_s\mathcal{M} \quad \textit{in } \ell^1(L^1((0,T)\times(a,b)\times\mathbb{R})) \quad \textit{and a.e.}$$

Proof We have proved in Lemma 4.2 the strong and a.e. convergence of N_s^η towards N_s. For this surface density $N_s \in L_t^\infty L_x^1$, we solve the Schrödinger-Poisson system at the equilibrium (1.19)–(1.20). It is proved in Proposition 3.1 of [34] that there exists a unique $V \in L^\infty([0,T],H^1(\Omega))$ solution of (1.19)–(1.20). We show hereinafter that the strong convergence in L^1 of the surface density allows to prove that

$$\|V^\eta - V\|_{L^2([0,T],H^1(\Omega))} \to 0 \quad \text{as } \eta \to 0. \tag{4.5}$$

In fact, we multiply the Poisson equation by $(V^\eta - V)$ and integrate, we have

$$\int_0^T \iint_\Omega |\nabla(V^\eta - V)|^2\, dx\, dz\, dt = I + II + III,$$

where

$$I = \sum_k \int_0^T \iint_{(a,b)\times\mathbb{R}} (f_k^\eta - N_s^\eta \mathcal{M}_k^\eta)\langle|\chi_k[V^\eta]|^2(V^\eta - V)\rangle\, dx\, dv\, dt,$$

$$II = \sum_k \int_0^T \iint_{(a,b)\times\mathbb{R}} N_s^\eta \mathcal{M}_k^\eta\langle(|\chi_k[V^\eta]|^2 - |\chi_k[V]|^2)(V^\eta - V)\rangle\, dx\, dv\, dt,$$

$$III = \sum_k \int_0^T \iint_{(a,b)\times\mathbb{R}} (N_s^\eta \mathcal{M}_k^\eta - N_s\mathcal{M}_k)\langle|\chi_k[V]|^2(V^\eta - V)\rangle\, dx\, dv\, dt.$$

We bound the first term thanks to the estimate on the dissipation rate (2.3). Lemma A.4 provides

$$|I| \le C_1 \sum_k \int_0^T \iint_{(a,b)\times\mathbb{R}} |f_k^\eta - N_s^\eta \mathcal{M}_k^\eta| e^{C_2\|V^\eta\|_{L_z^2}} \|V^\eta - V\|_{L_z^2}\, dx\, dv\, dt.$$

Thus Lemma 2.4 implies that

$$\begin{aligned} |I| &\le C_3 \sum_k \int_0^T \iint_{(a,b)\times\mathbb{R}} |f_k^\eta - N_s^\eta \mathcal{M}_k^\eta|\, dx\, dv\, dt \\ &\le 4C_3 \|N_s^\eta\|_{L_{t,x}^1}^{1/2} \left(\int_0^T \mathcal{R}^\eta(t)\, dt\right)^{1/2}, \end{aligned} \tag{4.6}$$

where we use the Cauchy-Schwarz inequality. Then Proposition 2.5 implies that $|I| \le C_4\eta$. For the second term we use the fact that the Maxwellian $\mathcal{M}_k$ decays with respect to k. Therefore using Lemma A.2, we deduce

$$II = \int_0^T \iint_{(a,b)\times\mathbb{R}} N_s^\eta \int_0^1 \frac{1}{2} \sum_{k,\ell\neq k} \frac{\mathcal{M}_k^\eta - \mathcal{M}_\ell^\eta}{\epsilon_k^\sigma - \epsilon_\ell^\sigma} \langle \chi_k^\sigma (V^\eta - V)\chi_\ell^\sigma\rangle^2\, d\sigma\, dx\, dv\, dt \le 0, \tag{4.7}$$

where we denote $\epsilon_k^\sigma := \epsilon_k[\sigma V + (1-\sigma)V^\eta]$ and $\chi_k^\sigma := \chi_k[\sigma V + (1-\sigma)V^\eta]$. Finally, the bound on the potential in $L^\infty([0,T], H^1(\Omega))$ combined with Lemma A.4 furnishes the estimate

$$|III| \le C_5 \int_0^T \|N_s\mathcal{M} - N_s^\eta \mathcal{M}^\eta\|_{\ell^1(L^1_{x,v})} \|V^\eta - V\|_{H^1_{x,z}}\, dt. \tag{4.8}$$

Moreover, using Lemma A.2, we can derive the function $s \mapsto e^{-\epsilon_k[sV+(1-s)V^\eta]}/\mathcal{Z}[sV+(1-s)V^\eta]$ and therefore obtain

$$\mathcal{M}_k^\eta - \mathcal{M}_k = \int_0^1 \frac{e^{-v^2/2}}{2\pi}\frac{e^{-\epsilon_k^s}}{\mathcal{Z}^s}\left(\frac{\sum_\ell \langle|\chi_\ell^s|^2(V^\eta - V)\rangle e^{-\epsilon_k^s}}{\mathcal{Z}^s} - \langle|\chi_k^s|^2(V^\eta - V)\rangle\right) ds,$$

where we use the notation $f^s := f[sV + (1-s)V^\eta]$. Then by Lemma A.1 we have

$$|\mathcal{M}_k^\eta - \mathcal{M}_k| \le C_1\, e^{C_2(\|V^\eta\|_{L^2_z} + \|V\|_{L^2_z})} \|V - V^\eta\|_{L^2_z} \int_0^1 \frac{e^{-v^2/2-\epsilon_k^s}}{2\pi\,\mathcal{Z}^s}\, ds.$$

Thus the Sobolev embedding $H^1(\Omega) \hookrightarrow L^\infty_x L^2_z(\Omega)$ provides

$$\sum_k \|\mathcal{M}_k^\eta - \mathcal{M}_k\|_{L^\infty_x L^1_v} \le C\|V^\eta - V\|_{H^1(\Omega)}. \tag{4.9}$$

This implies that

$$\begin{aligned}\|N_s\mathcal{M} - N_s^\eta\mathcal{M}^\eta\|_{\ell^1(L^1_{x,v})} &\le \|N_s - N_s^\eta\|_{L^1_x} + \|N_s^\eta(\mathcal{M} - \mathcal{M}^\eta)\|_{\ell^1(L^1_{x,v})}\\ &\le \|N_s - N_s^\eta\|_{L^1_x} + C\mathcal{N}_{in}\|V^\eta - V\|_{H^1(\Omega)},\end{aligned} \tag{4.10}$$

where we use (4.9) for the last inequality. Finally, from (4.8) we can bound the term III by

$$|III| \le C(\|N_s - N_s^\eta\|_{L^1_{t,x}} + \mathcal{N}_{in}\|V^\eta - V\|^2_{L^2([0,T],H^1(\Omega))}). \tag{4.11}$$

Thus, (4.6), (4.7) and (4.11) provide with the Poincaré inequality

$$\|V^\eta - V\|^2_{L^2([0,T],H^1(\Omega))} \le C(\eta + \|N_s - N_s^\eta\|_{L^1_{t,x}} + \mathcal{N}_{in}\|V^\eta - V\|^2_{L^2([0,T],H^1(\Omega))}).$$

Finally, if $\mathcal{N}_{in}$ is small enough, we deduce that $V^\eta \to V$ as $\eta \to 0$ in $L^2([0,T], H^1(\Omega))$.

The convergence of the distribution function f^η is yet obtained thanks to the estimate on the dissipation rate $\mathcal{R}^\eta$ (1.17). Then the properties of the eigenvalues of the Hamiltonian (A.2) and the embedding of $H^1(\Omega)$ into $L^\infty_x L^2_z(\Omega)$ (Lemma 2.4) show that

$$\|\epsilon_k[V^\eta] - \epsilon_k[V]\|_{L^2([0,T],L^\infty(a,b))} \le C\|V^\eta - V\|_{L^2([0,T],H^1(\Omega))} \to 0 \quad \text{as } \eta \to 0.$$

Moreover, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\|f^\eta - N_s\mathcal{M}\|_{\ell^1(L^1_{t,x,v})} &\le \|f^\eta - N_s^\eta\mathcal{M}^\eta\|_{\ell^1(L^1_{t,x,v})} + \|N_s^\eta\mathcal{M}^\eta - N_s\mathcal{M}\|_{\ell^1(L^1_{t,x,v})}\\ &\le 4\|N_s^\eta\|^{1/2}_{L^1_{t,x}}\left(\int_0^T \mathcal{R}^\eta(t)\, dt\right)^{1/2} + \|N_s^\eta\mathcal{M}^\eta - N_s\mathcal{M}\|_{\ell^1(L^1_{t,x,v})}.\end{aligned}$$

Thus the entropy inequality (2.3) and (4.10) yield that $f^\eta \to N_s\mathcal{M}$ strongly in $\ell^1(L^1_{t,x,v})$. □

4.2 The Limit Equation

To end the proof of Theorem 1.3, we have to prove that the limit N_s satisfies the drift-diffusion equation (1.18). Thanks to the local mass conservation (1.16), it suffices to study the limit of the current J^η.

Proposition 4.4 *Let (f^η, V^η) be a solution of the renormalized system defined in Theorem 1.2, then the current J^η, defined by*

$$J^\eta := \frac{1}{\eta}\sum_k \int_{\mathbb{R}^2} v f_k^\eta \, dv, \tag{4.12}$$

satisfies

$$\begin{cases} J^\eta \rightharpoonup J = -\mathbb{D}(\partial_x N_s + N_s \partial_x V_s) & \text{in weak-}L^1_{t,x}, \\ J(t,a) = J(t,b) = 0, \end{cases}$$

where the diffusion matrix $\mathbb{D}$ is defined in (2.2) and the autoconsistant potential is denoted

$$V_s = -\log\left(\sum_{k\geq 1} e^{-\epsilon_k[V]}\right).$$

The proof of this result is based on an idea of Masmoudi and Tayeb [21] consisting in using the point (i) of Theorem 1.2. Because of the dependence on k and of the non linear coupling, the proof is not straightforward. Then we detail the proof hereinafter.

Proof Thanks to Proposition 4.3 we have

$$\left(\sqrt{f_k^\eta}\right)_{k\geq 1} \to \left(\sqrt{N_s \mathcal{M}_k}\right)_{k\geq 1} \quad \text{in } \ell^2(L^2_{t,x,v}),$$

and the definition of r_k^η (4.1) implies

$$J^\eta = \frac{1}{\eta}\sum_k \int_{\mathbb{R}^2} v f_k^\eta \, dv = 2\sqrt{N_s^\eta}\sum_k \int_{\mathbb{R}^2} v r_k^\eta \mathcal{M}_k^\eta \, dv + \mathbf{O}(\eta)_{\ell^1(L^1_{t,x,v})}.$$

Besides, we have $\mathcal{M}_k^\eta \leq e^{-v^2/2-\pi^2 k^2/2}$ (see Appendix) and the bound (4.2) show that the sequence $(r^\eta \mathcal{M}^\eta / e^{-(v^2+\pi^2k^2)/4})_\eta$ is bounded in $\ell^2(L^2_{t,x,v})$. Thus up to an extraction, there is a u in $\ell^2(L^2_{t,x,v})$ such that $r^\eta \mathcal{M}^\eta / e^{-(v^2+\pi^2k^2)/4} \rightharpoonup u$ weakly in $\ell^2(L^2_{t,x,v})$. Setting $r = u\, e^{-(v^2+\pi^2k^2)/4}/\mathcal{M}$, we get that $(r^\eta \mathcal{M}^\eta / e^{-(v^2+\pi^2k^2)/4})$ weakly converges towards $(r\mathcal{M}/e^{-(v^2+\pi^2k^2)/4})$ in $\ell^2(L^2_{t,x,v})$. We deduce

$$\begin{aligned} \sum_k \int_{\mathbb{R}^2} v r_k^\eta \mathcal{M}_k^\eta \, dv &= \sum_k \int_{\mathbb{R}^2} v e^{-(v^2+\pi^2k^2)/4} \frac{r_k^\eta \mathcal{M}_k^\eta}{e^{-(v^2+\pi^2k^2)/4}} \, dv \\ &\rightharpoonup \sum_k \int_{\mathbb{R}^2} v r_k \mathcal{M}_k \, dv \quad \text{in weak-}L^2_{t,x}. \end{aligned}$$

Moreover, the strong convergence $\sqrt{N_s^\eta} \to \sqrt{N_s}$ in $L^2_{t,x}$ implies that

$$J^\eta \rightharpoonup J := 2\sqrt{N_s}\sum_k \int_{\mathbb{R}^2} v r_k \mathcal{M}_k \, dv \quad \text{in weak-}L^1_{t,x}. \tag{4.13}$$

Since we have $\sum_k \int v\mathcal{M}_k\, dv = 0$, Proposition 2.1 shows that we can define $Q^{-1}(v\mathcal{M})$ and the selfadjointness of the operator Q leads to

$$J = 2\sqrt{N_s}\sum_k \int_{\mathbb{R}^2} Q^{-1}(v\mathcal{M})_k Q(r\mathcal{M})_k \frac{dv}{\mathcal{M}_k}. \tag{4.14}$$

Now, we will find an expression of J. Considering again r_k^η, we have

$$\frac{Q^\eta(f^\eta)_k}{\eta} = 2\sqrt{N_s^\eta}\, Q^\eta(r^\eta\mathcal{M}^\eta)_k + \eta Q^\eta((r^\eta)^2\mathcal{M}^\eta)_k.$$

With (4.2), the second term in the right hand side is $\mathbf{O}(\eta)_{\ell^1(L^1_{t,x,v})}$. For the first one, one can prove easily that $\forall f \in \ell^2(L^2_{t,x,v})$, we have $\|Q^\eta(f) - Q(f)\|_{\ell^2(L^2_{t,x,v})} \to 0$. The weak convergence of $(r^\eta\mathcal{M}^\eta)$ in $\ell^2(L^2_{t,x,v})$ implies then

$$Q^\eta(r^\eta\mathcal{M}^\eta) \rightharpoonup Q(r\mathcal{M}) \quad \textit{weakly} \text{ in } \ell^2(L^2_{t,x,v}).$$

With the strong convergence in $L^2_{t,x}$ of $\sqrt{N_s^\eta}$, we deduce the weak limit:

$$\begin{aligned}\frac{Q^\eta(f^\eta)_k}{\eta} &= 2\sqrt{N_s^\eta\mathcal{M}_k}\frac{Q^\eta(r^\eta\mathcal{M}^\eta)_k}{\sqrt{\mathcal{M}_k}} + \mathbf{O}(\eta)_{\ell^1(L^1_{t,x,v})} \\ &\rightharpoonup 2\sqrt{N_s}\, Q(r\mathcal{M})_k \quad \text{in } \ell^1(L^1_{t,x,v}).\end{aligned} \tag{4.15}$$

We recall that for every $\lambda > 0$, we have defined $\Theta^\eta_{k,\lambda} = (f_k^\eta + \lambda\exp(-\frac{1}{2}(v^2+k^2)))^{1/2}$. Thus

$$\Theta^\eta_{k,\lambda} \to \Theta_{k,\lambda} = (f_k + \lambda e^{-\frac{1}{2}(v^2+k^2)})^{1/2} \quad \text{strongly in } \ell^2(L^2_{t,x,v}).$$

Using Lemmata A.1 and A.2 and the estimate (A.3), we can prove that

$$\begin{aligned}|\partial_x\epsilon_k^\eta - \partial_x\epsilon_k| &\le |\langle(|\chi_k^\eta|^2 - |\chi_k|^2)\partial_x V^\eta\rangle| + |\langle|\chi_k|^2\partial_x(V^\eta - V)\rangle| \\ &\le C_1 e^{C_2\|V\|_{L^2_z}}(e^{C_2\|V^\eta\|_{L^2_z}}\|\partial_x V^\eta\|_{L^2_z}\|V - V^\eta\|_{L^2_z} + \|\partial_x(V-V^\eta)\|_{L^2_z}).\end{aligned}$$

Thus the strong convergence of V^η in $L^2_t(H^1_{x,z})$ with the Sobolev embedding $H^1(\Omega) \hookrightarrow L^\infty_x L^2_z(\Omega)$ imply that

$$\partial_x\epsilon_k^\eta \to \partial_x\epsilon_k \quad \text{strongly in } \ell^2(L^2_{t,x}).$$

Therefore,

$$\partial_x\epsilon_k^\eta\,\Theta^\eta_{k,\lambda} \rightharpoonup \partial_x\epsilon_k\,\Theta_{k,\lambda} \quad \textit{weakly} \text{ in } L^1_{t,x,v}.$$

Thus we can take the weak limit as $\eta \to 0$ in (1.15). For all $k \ge 1$, we find

$$v\cdot\partial_x\Theta_{k,\lambda} - \partial_v(\partial_x\epsilon_k\,\Theta_{k,\lambda}) = \frac{\sqrt{N_s}\,Q(r\mathcal{M})_k}{\Theta_{k,\lambda}} + \lambda\partial_x\epsilon_k\frac{ve^{-\frac{1}{2}(v^2+k^2)}}{2\Theta_{k,\lambda}}.$$

And if we make $\lambda \to 0$ in the resulting equation, we find

$$\left(\partial_x\sqrt{N_s} + \frac{1}{2}\sqrt{N_s}\partial_x V_s\right)\cdot v\mathcal{M}_k = Q(r\mathcal{M})_k, \tag{4.16}$$

where we take $V_s = -\log\sum_k e^{-\epsilon_k}$. And we verify that the product $\sqrt{N_s}\partial_x V_s$ has a meaning in $L^1_{t,x}$. Now, with (4.13) and (4.16), we can conclude

$$J = -\mathbb{D}\sqrt{N_s}\left(\partial_x\sqrt{N_s} + \frac{1}{2}\sqrt{N_s}\partial_x V_s\right), \tag{4.17}$$

where the symmetric positive diffusion matrix, defined in (2.2), is given by

$$\mathbb{D} = -\sum_k \int_{\mathbb{R}} v \otimes Q^{-1}(v\mathcal{M})_k \, dv.$$

Besides, with our choice of boundary conditions (1.8) we have that

$$\sum_{k\geq 1}\int_{\mathbb{R}} v f_k^\eta(t,a,v)\,dv = \sum_{k\geq 1}\int_{\mathbb{R}} v f_k^\eta(t,b,v)\,dv = 0.$$

Thus as η goes to 0, it provides that $J(t,a) = 0$ and $J(t,b) = 0$. Now, if we use Lemma 4.5 combined with (4.16), we can rewrite the current J and the proof of Proposition 4.4 is complete. □

Lemma 4.5 *Let N_s and V be defined in Proposition 4.3. If we suppose that*

$$\partial_x\sqrt{N_s} + \frac{1}{2}\sqrt{N_s}\partial_x V_s = G \in L^2((0,T)\times(a,b)), \tag{4.18}$$

where $V_s = -\log(\sum_{k\geq 1} e^{-\epsilon_k[V]})$. Then we have

$$\sqrt{N_s} \in L^2((0,T), H^1(a,b)) \quad \textit{and} \quad \sqrt{N_s}\partial_x V_s \in L^2((0,T)\times(a,b)).$$

Proof We have $\sqrt{N_s}$ bounded in $L^2_{t,x}$ and V in $L^2_t H^1_x$, then from Lemma A.1, we deduce that $\sqrt{N_s}\partial_x V_s \in L^1_{t,x}$. It follows that $\partial_x\sqrt{N_s} \in L^1_{t,x}$. We consider the approximation of the identity β_δ as before. Namely $\beta_\delta(s) = \frac{1}{\delta}\beta(\delta s)$ where β is a $C^\infty(\mathbb{R}^+)$ function satisfying $\beta(s) = s$ for $0 \leq s \leq 1$, $\beta(s) = 2$ for $s \geq 3$ and $0 \leq \beta'(s) \leq 1$. If we denote $\psi = \sqrt{N_s}$, we have

$$\partial_x\beta_\delta(\psi) = \partial_x\psi\beta'_\delta(\psi).$$

Hence we can renormalize (4.18):

$$\partial_x\beta_\delta(\psi) + \frac{1}{2}\partial_x V_s\,\beta'_\delta(\psi)\psi = \tilde{G}$$

where $\tilde{G} = G\,\beta'_\delta(\psi) \leq G$. Multiplying (4.19) by $\partial_x\beta_\delta(\psi)$ and integrating provides

$$\iint |\partial_x\beta_\delta(\psi)|^2\,dx\,dt + \frac{1}{2}\iint \partial_x V_s \cdot \partial_x\beta_\delta(\psi)\,\psi\beta'_\delta(\psi)\,dx\,dt$$
$$= \iint \tilde{G}\partial_x\beta_\delta(\psi)\,dx\,dt. \tag{4.19}$$

By the Cauchy-Schwarz inequality we deduce

$$\iint \tilde{G}\partial_x\beta_\delta(\psi)\,dx\,dt \leq \frac{1}{2}\iint \tilde{G}^2\,dxdt + \frac{1}{2}\iint |\partial_x\beta_\delta(\psi)|^2\,dx\,dt.$$

If we define $\tilde\beta$ by $\tilde\beta(s) = \int_0^s \tau\beta'(s)^2\,d\tau$ and $\tilde\beta_\delta(s) = \frac{1}{\delta^2}\tilde\beta(\delta s)$. Then, $\tilde\beta_\delta(s)$ tends to $\frac{s^2}{2}$ when δ goes to 0 and we have

$$\int_a^b \partial_x V_s \cdot \partial_x \beta_\delta(\psi)\,\psi\beta_\delta'(\psi)\,dx = \int_a^b \partial_x V_s \cdot \partial_x \tilde\beta_\delta(\psi)\,dx = -\int_a^b \partial_x^2 V_s\,\tilde\beta_\delta(\psi)\,dx. \tag{4.20}$$

Thanks to the Poisson equation (1.20), we have:

$$\begin{aligned}-\partial_x^2 V_s = &-4\sum_k \frac{e^{-\epsilon_k}(\epsilon_k)^2}{\mathcal{Z}} + \frac{\langle N^2 + 4V^2 N\rangle}{N_s} + 2\sum_k \frac{e^{-\epsilon_k}}{\mathcal{Z}}\langle (V+\epsilon_k)|\partial_z\chi_k|^2\rangle \\ &- \frac{1}{\mathcal{Z}}\sum_k\sum_{\ell\neq k}\left(\frac{e^{-\epsilon_k} - e^{-\epsilon_\ell}}{\epsilon_k - \epsilon_\ell}\right)\langle \chi_k\chi_\ell\,\partial_x V\rangle^2 \\ &+ \sum_k \frac{e^{-\epsilon_k}}{\mathcal{Z}}\langle |\chi_k|^2\,\partial_x V\rangle^2 - \left(\sum_k \frac{e^{-\epsilon_k}}{\mathcal{Z}}\langle |\chi_k|^2\,\partial_x V\rangle\right)^2.\end{aligned} \tag{4.21}$$

By the Cauchy-Schwarz inequality, the sum of the last two terms of the right hand side is nonnegative. Moreover, except for the first one, the other terms are obviously nonnegative. Thus we have with (4.20),

$$\int_a^b \partial_x V_s \cdot \partial_x \beta_\delta(\psi)\,\psi\beta_\delta'(\psi)\,dx \ge -4\int_a^b \sum_k \frac{e^{-\epsilon_k}(\epsilon_k)^2}{\mathcal{Z}}\tilde\beta_\delta(\psi)\,dx.$$

Moreover, Lemma A.1 and the Sobolev embedding $H^1(\Omega) \hookrightarrow L_x^\infty L_z^2(\Omega)$ imply that $\sum_k \frac{e^{-\epsilon_k}(\epsilon_k)^2}{\mathcal{Z}}$ is bounded in $L^\infty(a,b)$. Thus (4.19) leads to

$$\iint |\partial_x \beta_\delta(\psi)|^2\,dt\,dx \le \iint G^2\,dt\,dx + 4\int_a^b \sum_k \frac{e^{-\epsilon_k}(\epsilon_k)^2}{\mathcal{Z}}\tilde\beta_\delta(\psi)\,dx.$$

Passing to the limit $\delta \to 0$, we have

$$\iint |\partial_x\sqrt{N_s}|^2\,dt\,dx \le \iint G^2\,dt\,dx + 4\int_a^b \sum_k \frac{e^{-\epsilon_k}(\epsilon_k)^2}{\mathcal{Z}} N_s\,dx.$$

Thus we deduce that $\sqrt{N_s} \in L^2((0,T), H^1(a,b))$ and with (4.18) we conclude easily that $\sqrt{N_s}\partial_x V_s \in L^2((0,T)\times(a,b))$. □

5 Existence for the Overall Problem

5.1 The Truncated Boltzmann Equation

This part deals with well-known existence results and properties for the Boltzmann equation. The results will be given for the matter of completeness without proof, we refer to [2, 4, 8, 11, 23] for more details. We shall assume that $\eta > 0$ is fixed, for the clarity of the notation we chose $\eta = 1$, and that the force fields $F_k := -\partial_x\epsilon_k$ is given. We consider the Boltzmann equations indexed in k:

$$\begin{cases} \partial_t f_k + v \cdot \partial_x f_k + F_k \cdot \partial_v f_k = Q_R(f)_k, & (x,v) \in (a,b)\times\mathbb{R},\ t\in[0,T], \\ f_k(t,a,v) = f_k(t,a,-v), \qquad f_k(t,b,v) = f_k(t,b,-v) & \text{for } t\in[0,T],\ v>0, \\ f_k(0,x,v) = f_k^{in}(x,v), \end{cases} \tag{5.1}$$

with the truncated collision operator:

$$Q_R(f)_k = \sum_{k'} \int_{\mathbb{R}} \alpha^R_{k,k'}(v,v')(\mathcal{M}_k(v) f_{k'}(v') - \mathcal{M}_{k'}(v') f_k(v))\,dv', \tag{5.2}$$

where the truncated cross-section is defined for a $R>0$ by

$$\alpha^R_{k,k'}(v,v') = \alpha_{k,k'}(v,v') 1_{k\le R, |v|\le R}(k,v) 1_{k'\le R,|v'|\le R}(k',v'). \tag{5.3}$$

A simple calculation shows that the regularized collision operator (5.2) is bounded in $\ell^1(L^1_{t,x})$ and in $\ell^\infty(L^\infty_{t,x})$ and satisfies $\sum_k \int_{\mathbb{R}} Q_R(f)_k\,dv = 0$.

We can prove, using the characteristics techniques (see for instance [4, 6, 8]), existence and uniqueness of weak solutions for each equation of (5.1):

Lemma 5.1 *Let $T>0$ and assume that the initial data satisfy for all $k\ge 1$,*

$$f_k^{in} \ge 0, \quad (1+v^2) f_k^{in} \in L^1((a,b)\times\mathbb{R}), \qquad f_k^{in} \in L^\infty((a,b)\times\mathbb{R}).$$

Assume that $F_k \in L^1((0,T), W^{1,1}(a,b)\cap L^\infty(a,b))$ and that $\epsilon_k \ge \frac{1}{2}\pi^2 k^2$. Then (5.1) admits a unique weak solution $f_k \in L^\infty((0,T), L^1\cap L^\infty((a,b)\times\mathbb{R}))$, $f_k \ge 0$ and

$$\forall t\in[0,T], \quad \sum_{k\ge1} \int_{\mathbb{R}} f_k(t,x,v)\,dv = \sum_{k\ge1}\int_{\mathbb{R}} f_k^{in}(x,v)\,dv. \tag{5.4}$$

Moreover if there exists $\delta>2$ such that $(v^\delta + k^2) f_k^{in} \in \ell^\infty(L^\infty_{x,v})$ then $\forall t\in[0,T]$,

$$\sum_{k\ge1} \|f_k(t,\cdot,\cdot)\|_{L^\infty((a,b)\times\mathbb{R})} \le C\left(1 + \left(\int_0^t \sup_{k\ge1} \|F_k(s,\cdot)\|_{L^\infty_x}\,ds\right)^2\right), \tag{5.5}$$

where C is a constant depending only on T and the data.

5.2 Proof of Theorem 1.2

In this section we give the sketch of the proof of Theorem 1.2. The structure of the coupling invite us to use a fixed-point argument for the proof. However to define this fixed-point, the uniqueness of a solution of the Schrödinger-Poisson system is needed. Thus we are not able to prove the existence for every kind of initial condition but only for small initial data.

The main steps for the proof, described hereinafter, follow the idea of [2, 4, 21]: we regularize the system thanks to a small parameter $\varepsilon>0$, we construct solution of the regularized system and we left go the parameter ε to 0 to recover solutions of the unregularized system.

First, let us define the linear regularization operator by

$$\begin{aligned} &R^\varepsilon : L^1(\Omega) \to C^\infty(\overline{\Omega}), \\ &\qquad V \to R^\varepsilon[V](x,z) = (\overline{V} *_x \xi_{\varepsilon,x} *_z \xi_{\varepsilon,z})|_{\overline{\Omega}} \end{aligned} \tag{5.6}$$

where $\overline{V}$ is the extension of V by zero outside Ω and $\xi_{\varepsilon,x}$ and $\xi_{\varepsilon,z}$ are C^∞ nonnegative compactly supported even approximations of the unity on $\mathbb{R}$. Moreover, we can prove straightforwardly from convolution results that the regularization operator R^ε satisfies the following properties:

Lemma 5.2

(i) *R^ε is a bounded operator on $L^p_x L^q_z(\Omega)$ for $1 \le p, q \le +\infty$ and satisfies for all $V \in L^p_x L^q_z(\Omega)$,*

$$\|R^\varepsilon[V]\|_{L^p_x L^q_z(\Omega)} \le \|V\|_{L^p_x L^q_z(\Omega)} \quad \textit{and} \quad \lim_{\varepsilon\to 0} \|R^\varepsilon[V] - V\|_{L^p_x L^q_z(\Omega)} = 0.$$

(ii) *R^ε is self-adjoint on $L^2(\Omega)$ and for all $V \in W^{1,2}(\Omega)$,*

$$\nabla_x R^\varepsilon[V] = R^\varepsilon[\nabla_x V]; \quad \lim_{\varepsilon\to 0} \|\nabla_x R^\varepsilon[V] - \nabla_x V\|_{L^2(\Omega)} = 0.$$

We introduce then the regularized system:

$$\begin{cases} \partial_t f^\varepsilon_{k,R} + \frac{1}{\eta}(v \cdot \partial_x f^\varepsilon_{k,R} - \partial_x \epsilon^\varepsilon_{k,R} \cdot \partial_v f^\varepsilon_{k,R}) = \frac{1}{\eta^2} Q^\varepsilon_R(f^\varepsilon_R)_k, \quad (x,v) \in (a,b)\times\mathbb{R}, \\ f^\varepsilon_{k,R}(t,a,v) = f^\varepsilon_{k,R}(t,a,-v), \qquad f^\varepsilon_{k,R}(t,b,v) = f^\varepsilon_{k,R}(t,b,-v), \quad v>0, \\ f^\varepsilon_{k,R}(0,x,v) = f^{in}_k(x,v), \end{cases} \tag{5.7}$$

$$\begin{cases} -\frac{1}{2}\partial_z^2 \chi^\varepsilon_{k,R} + R^\varepsilon[V^\varepsilon_R]\chi^\varepsilon_{k,R} = \epsilon^\varepsilon_{k,R}\chi^\varepsilon_{k,R} \quad (k \ge 1), \\ \chi^\varepsilon_{k,R}(t,x,\cdot) \in H^1_0(0,1), \qquad \int_0^1 \chi^\varepsilon_{k,R}\chi^\varepsilon_{\ell,R}\,dz = \delta_{k\ell}, \end{cases} \tag{5.8}$$

$$\begin{cases} -\Delta_{x,z} V^\varepsilon_R = R^\varepsilon[\sum_k \int_{\mathbb{R}} f^\varepsilon_{k,R}|\chi^\varepsilon_{k,R}|^2\,dv], \\ \frac{dV^\varepsilon_R}{dx}(t,a,z) = \frac{dV^\varepsilon_R}{dx}(t,b,z) = 0, \quad \text{for } z \in (0,1), \\ V^\varepsilon_R(t,x,0) = V^\varepsilon_R(t,x,1) = 0, \quad \text{for } x \in (a,b). \end{cases} \tag{5.9}$$

We use the regularization of the collision operator:

$$Q^\varepsilon_R(f)_k = \sum_{k'} \int_{\mathbb{R}} \alpha^R_{k,k'}(v,v')(\mathcal{M}^\varepsilon_k(v) f_{k'}(v') - \mathcal{M}^\varepsilon_{k'}(v') f_k(v))\,dv', \tag{5.10}$$

where the truncated cross-section is defined for $R > 0$ in (5.3). We use the notations of Sect. 1:

$$N^\varepsilon_s = \sum_{k\ge 1}\int_{\mathbb{R}} f^\varepsilon_k\,dv \quad \text{and} \quad \mathcal{M}^\varepsilon_k = \frac{1}{2\pi\mathcal{Z}^\varepsilon}\exp\left(-\frac{1}{2}v^2 - \epsilon^\varepsilon_k\right) \quad \text{for } \mathcal{Z}^\varepsilon = \sum_{k\ge 1} e^{-\epsilon^\varepsilon_k}.$$

Since for $\varepsilon = 0$ we have $R^0 = Id$, we will obtain a solution of the unregularized system by passing to the limits $\varepsilon \to 0$ and $R \to +\infty$ in the regularized one (5.7)–(5.9). Therefore the proof of Theorem 1.2 can be split in the three followings steps:

Step 1: Existence for the regularized problem. In the first step we prove that the regularized problem admits a solution. We verify easily that the regularized collision operator (5.2) is bounded in $\ell^1(L^1_{t,x})$ and in $\ell^\infty(L^\infty_{t,x})$ and satisfies $\sum_k \int_{\mathbb{R}} Q^\varepsilon_R(f^\varepsilon)_k\,dv = 0$ and

$$\sum_{k\ge 1}\int_{\mathbb{R}} Q^\varepsilon_R(f^\varepsilon)_k \log\frac{f^\varepsilon_k}{\mathcal{M}^\varepsilon_k}\,dv \le -\frac{\alpha_1}{2}\sum_{k\ge 1}\int_{\mathbb{R}}\left(\sqrt{f^\varepsilon_k} - \sqrt{N^\varepsilon_s\mathcal{M}^\varepsilon_k}\right)^2 dv.$$

Following the ideas of the proof of Proposition 4.8 of [4] we establish:

Proposition 5.3 *Let* $T > 0$ *and let assume that Assumption* (A.1) *holds and that the initial condition is at the thermal equilibrium, i.e. verify* (A.2) *and is given by* (1.9). *Then, there exists* $\varepsilon_0 > 0$ *and* $\delta > 0$ *such that, if*

$$\sum_{k\geq 1} \|f_k^{in}\|_{L^1_{x,v}} < \delta, \tag{5.11}$$

and $\varepsilon \in (0, \varepsilon_0)$ *then the regularized problem* (5.7)–(5.9) *admits a global weak solution* $(V_R^\varepsilon, (f_{k,R}^\varepsilon)_{k\geq 1})$ *on the interval* $[0, T]$ *which satisfies the entropy estimate*:

$$\forall t \in [0, T], \quad 0 \leq W_R^\varepsilon(t) + \frac{\alpha_1}{\eta^2}\int_0^T \mathcal{R}_R^\varepsilon(t)\, dt \leq C_T, \tag{5.12}$$

with

$$W_R^\varepsilon(t) = \sum_{k\geq 1}\left(f_{k,R}^\varepsilon \log \frac{f_{k,R}^\varepsilon}{M_k} - f_{k,R}^\varepsilon + M_k\right) dx\, dv + \frac{1}{2}\int\!\!\int |\nabla_{x,z} V_R^\varepsilon|^2\, dx\, dz$$

and

$$\mathcal{R}_R^\varepsilon(t) = \frac{1}{2}\sum_{k\geq 1}\int\!\!\int \left(\sqrt{f_{k,R}^\varepsilon} - \sqrt{N_{s,R}^\varepsilon \mathcal{M}_{k,R}^\varepsilon}\right)^2 dx\, dv.$$

Step 2: Passing to the limit $R \to +\infty$. For $\varepsilon > 0$ fixed, one can pass to the limit as $R \to +\infty$. We obtain

Proposition 5.4 *Let* $T > 0$ *and let assume that* (A.1) *and* (A.2) *are satisfied. Let* $\varepsilon > 0$ *be fixed* ($\varepsilon < \varepsilon_0$) *and* $(V_R^\varepsilon, (f_{k,R}^\varepsilon, \chi_{k,R}^\varepsilon, \epsilon_{k,R}^\varepsilon)_{k\geq 1})$ *be a weak solution of the regularized Boltzmann-Schrödinger-Poisson system* (5.7)–(5.9). *Then as* $R \to +\infty$ *this solution converges to a weak solution* $(V^\varepsilon, (f_k^\varepsilon, \chi_k^\varepsilon, \epsilon_k^\varepsilon)_{k\geq 1})$ *of the regularized Boltzmann-Schrödinger-Poisson system* (5.7)–(5.9) *with* Q_R^ε *is substituted by* Q_R^ε *in the Boltzmann equation* (5.7).

Moreover it satisfies the entropy estimate (5.12) *with* f_k^ε *instead of* $f_{k,R}^\varepsilon$.

Proof We skip all the index ε in the notation. With our regularization (5.6) we have a bound on V in $L_t^\infty(W_{x,z}^{1,\infty})$ depending only on ε but not on R. It provides thanks to (5.5) a bound on $(f_{k,R})_{k\geq 1}$ in $\ell^\infty(L_{t,x,v}^\infty)$ depending only on ε and on the data. And with (5.4), we have a bound on $(f_{k,R})_{k\geq 1}$ in $\ell^1(L_{t,x,v}^1)$ depending only on the data. Thus we can extract a subsequence converging as $R \to +\infty$ towards a function f in $\ell^2(L_{t,x,v}^2)$-*weak*. Using the standard mean compactness result (see Theorem 1.8 of [8], see also [16]), we deduce the relative strong compactness of the sequence indexed by R

$$\int_{\mathbb{R}} f_{k,R}\psi_k\, dv$$

in $L_{loc}^2([0, T] \times (a, b))$ for all $\psi_k \in \mathcal{D}(\mathbb{R})$ all null except for a finite number of them. Using the fact that the quantity $(1 + k^2) f_{k,R}$ is bounded in $L_t^\infty(\ell^1(L_{x,v}^1))$ we can choose $\psi_k = 1$. Thus one obtain that $\rho_R := (\int f_{k,R}\, dv)_{k\geq 1} \to \rho := (\int f_k\, dv)_{k\geq 1}$ in $L_{t,x}^2$-*strong*.

The conservation of the mass implies that for all $t \in [0,T]$ we have $\|f\|_{\ell^1(L^1_{t,x,v})} = \|f^{in}\|_{\ell^1(L^1_{t,x,v})} = \mathcal{N}_{in}$. Then we can solve the regularized Schrödinger-Poisson system (5.8)–(5.9) with the given $\rho := \int f\, dv$ and construct a unique solution $V \in L^\infty_t(H^1_{x,z})$. Using the fact that the sequence $(f_R)_R$ satisfies (5.12), we can use the continuity property of the solution of the Schrödinger-Poisson system (cf. Proposition 3.4) to prove that the sequence $(V_R)_R$ is Cauchy and therefore converges towards V in $L^2_t(H^1_{x,z})$. Properties of the eigenvalues of the Hamiltonian show that $\epsilon_k[R^\varepsilon[V_R]] \to \epsilon_k[R^\varepsilon[V]]$ in $L^2_t(W^{2,\infty}_{x,z})$.

Furthermore, we have for all $k \geq 1$

$$\|Q_R(f_R)_k\|_{L^\infty_{x,v}} \leq \alpha_2(\|f_R\|_{\ell^1(L^1_{x,v})} + \|f_R\|_{\ell^\infty(L^\infty_{x,v})}) \leq C_{T,\varepsilon},$$

where $C_{T,\varepsilon}$ is a nonnegative constant depending only on T and ε and on the data. We deduce that we can extract a subsequence $(Q_R(f_R)_k)_R$ converging as $R \to +\infty$ in L^∞-*weak**. Then from the definition of Q_R (5.2), we deduce that

$$\forall \phi \in L^1((a,b)\times\mathbb{R}), \qquad \iint (Q(f)_k - Q_R(f_R)_k)\phi\, dx\, dv \to 0 \quad \text{as } R \to +\infty.$$

Thus one can pass to the limit in the weak formulation of the Boltzmann-Schrödinger-Poisson system (5.7)–(5.9) and prove straightforwardly that $(V, (f_k, \epsilon_k, \chi_k)_{k\geq 1})$ is a solution of (5.7)–(5.9) with Q instead of Q_R. Finally we recover the entropy estimate by passing to the limit $R \to +\infty$ in (5.12). □

Step 3: Passing to the limit $\varepsilon \to 0$. In the last step we prove Theorem 1.2 by taking the limit $\varepsilon \to 0$.

Since the solution satisfies the entropy estimate, we deduce that

$$\sum_{k\geq 1} \iiint_{(0,T)\times(a,b)\times\mathbb{R}} f_k^\varepsilon (1 + v^2 + k^2 + |\log f_k^\varepsilon|)\, dx\, dv\, dt \leq C_T.$$

Thus the Dunford-Pettis Theorem and the De La Vallée Poussin Theorem implies that $(f_k^\varepsilon)_{k\geq 1}$ and is weakly relatively compact respectively in $\ell^1(L^1((0,T)\times(a,b)\times\mathbb{R}))$. Using standard mean compactness result (see e.g. Theorem 1.8 of [8]), we deduce the strong relative compactness of the sequence $(\rho_k^\varepsilon)_\varepsilon$ in $L^1([0,T]\times(a,b))$. Therefore, up to an extraction, we have

$$\rho_k^\varepsilon \to \rho_k \quad \text{strongly in } \ell^1(L^1((0,T)\times(a,b))). \tag{5.13}$$

Moreover ρ satisfies the estimate

$$\sum_{k\geq 1} \iint \rho_k (1+k^2)\, dx\, dt \leq C_T \tag{5.14}$$

and the conservation of the mass implies

$$\forall t \in [0,T], \forall \varepsilon > 0, \qquad \int_a^b N_s\, dx = \int_a^b N_s^\varepsilon\, dx = \mathcal{N}_{in}.$$

We can then apply Lemma 3.2 to solve the *unregularized* Schrödinger-Poisson system (3.1)–(3.2) for the density ρ and construct $V \in L^\infty([0,T], H^1(\Omega))$ which is unique thanks to

Lemma 3.3. Moreover multiplying the two Poisson equations by $(V^\varepsilon - V)$ and integrating lead to

$$\iint_\Omega |\nabla(V^\varepsilon - V)|^2\,dx\,dz = \iint_\Omega R^\varepsilon\Big[\sum_k(\rho_k^\varepsilon|\chi_k^\varepsilon|^2 - \rho_k|\chi_k|^2)\Big](V^\varepsilon - V)\,dx\,dz + \iint (R^\varepsilon - Id)\Big[\sum_k \rho_k|\chi_k|^2\Big](V^\varepsilon - V)\,dx\,dz. \quad (5.15)$$

Using the fact that with Lemma 5.2, $\|R^\varepsilon - Id\|_2 \to 0$ as $\varepsilon \to 0$, where

$$\|R^\varepsilon - Id\|_2 := \sup_{\{V\in L^2(\Omega), V\neq 0\}} \frac{\|(R^\varepsilon - Id)V\|_{L^2(\Omega)}}{\|V\|_{L^2(\Omega)}},$$

then we can prove, adapting the techniques of Proposition 3.4 that

$$\iint_\Omega |\nabla(V^\varepsilon - V)|^2\,dx\,dz \le C_1\|R^\varepsilon - Id\|_2\,\|V^\varepsilon - V\|_{H^1(\Omega)} + C_2\|\rho_k^\varepsilon - \rho_k\|_{\ell^1(L_x^1)}\,\|V^\varepsilon - V\|_{H^1(\Omega)} + C_3\mathcal{N}_{in}\|V^\varepsilon - V\|^2_{H^1(\Omega)}.$$

With the Poincaré inequality, we have for $\mathcal{N}_{in}$ small enough,

$$\|V^\varepsilon - V\|_{H^1(\Omega)} \le C(\|R^\varepsilon - Id\|_2 + \|\rho_k^\varepsilon - \rho_k\|_{\ell^1(L_x^1)}).$$

Thus there exists $\mathcal{N}_0 > 0$ such that, for all $0 < \mathcal{N}_{in} \le \mathcal{N}_0$, there exists $V \in L^\infty([0,T], H^1(\Omega))$ weak solution of the *unregularized* Schrödinger-Poisson system (3.1)–(3.2) and such that the potential V^ε, weak solution of the regularized system, converges towards V in $L^2([0,T], H^1(\Omega))$. The properties of the eigenvectors imply (see proof of Proposition 4.3) that $\epsilon_k^\varepsilon \to \epsilon_k$ in $L_t^2(L_x^\infty)$.

The end of the proof of Theorem 1.2 is standard (see [18, 21, 23]) and is based on a double renormalization. We first write the equation satisfied by $\beta_\delta(f^\varepsilon)$ with the function β_δ defined in Sect. 4.1 and weakly pass to the limit $\varepsilon \to 0$. Then we renormalize the resulting limit equation by β and let finally δ going to 0.

Remark 5.5 The convergence of the potential V^ε is a key point in this proof of existence. We notice that the technique used here relies strongly on the embedding $H^1(\Omega) \hookrightarrow L_x^\infty L_z^2(\Omega)$ which is not true when the x-variable is two dimensional. Then in this latter case we are not able to prove uniqueness of solutions of the Schrödinger-Poisson system for $(\rho_k)_k$ given and therefore the fixed point procedure does not converge. Thus the techniques used here do not allow us to prove the existence of solution of the coupled kinetic-quantum model for a two dimensional transport direction. However in the diffusive regime, the occupation factor ρ_k decays with respect to k and it has been proved in [34] that this allows us to recover uniqueness of solutions of the Schrödinger-Poisson system (in fact we can show in this case that the last term in (3.11) is nonpositive). Using the Trudinger estimate for the entropy functional furnishes existence of solutions of the drift-diffusion-Schrödinger-Poisson system (see [34]).

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Appendix: Spectral Properties of the Hamiltonian

In this appendix, we list some basic properties of eigenfunctions and eigenvalues of the Schrödinger operator in the z variable. For a given real valued function V in $L^2(0,1)$, let $H[V]$ be the Schrödinger operator

$$H[V] := -\frac{1}{2}\frac{d^2}{dz^2} + V(z)$$

defined on the domain $D(H[V]) = H^2(0,1) \cap H_0^1(0,1)$. This operator admits a strictly increasing sequence of real eigenvalues $(\epsilon_k[V])_{k\geq 1}$ going to $+\infty$. The corresponding eigenvectors, denoted by $(\chi_k[V](z))_{k\geq 1}$ (chosen such that $\chi_k'(0) > 0$ and $\int_0^1 |\chi_k[V]|^2\,dz = 1$), form an orthonormal basis of $L^2(0,1)$. They satisfy of course

$$-\frac{1}{2}\frac{d^2}{dz^2}\chi_k + V\chi_k = \epsilon_k\chi_k, \quad \chi_k \in H_0^1(0,1),\ \forall k \geq 1. \tag{A.1}$$

Obviously, for $V = 0$, we have $\epsilon_k[0] = \frac{1}{2}\pi^2 k^2$ and $\chi_k[0](z) = \sqrt{2}\sin(\pi k z)$. And

$$\text{if } U \leq V \text{ a.e. in } (0,1) \quad \text{then } \forall k \geq 1, \quad \epsilon_k[U] \leq \epsilon_k[V].$$

In the sequel we will use the standard notation $\langle f\rangle = \int_0^1 f(z)\,dz$ and when there is no confusion possible ϵ_k will stand for $\epsilon_k[V]$ and χ_k for $\chi_k[V]$. Following the study of the spectral properties of $H[V]$ in Chap. 2 of [28], we have:

Lemma A.1 *There exists a positive constant C_V depending only on $\|V\|_{L^2(0,1)}$ such that*

$$\left|\epsilon_k[V] - \frac{1}{2}\pi^2 k^2\right| \leq C_V; \qquad \|\chi_k[V] - \sqrt{2}\,\sin(\pi k z)\|_{L^\infty(0,1)} \leq C_V.$$

Moreover the constant C_V can be chosen such that $C_V \leq C_1 \exp(C_2\|V\|_{L^2(0,1)})$, where the constants C_1 and C_2 are independent of V and k.

Lemma A.2 *Let $V = V(\lambda, z) \in L^\infty_{loc}(0,\Lambda; L^2_z(0,1))$ with $\lambda \in (0,\Lambda)$ (typically $\lambda = t$ or $\lambda = x_i$). If $\partial_\lambda V \in L^1_{loc}(\lambda, L^2_z(0,1))$, then $\partial_\lambda\epsilon_k \in L^1_{loc}$, $\partial_\lambda\chi_k \in L^1_{loc}(\lambda, L^\infty_z(0,1))$ and we have*

$$\partial_\lambda\epsilon_k = \langle|\chi_k|^2\partial_\lambda V\rangle \quad \textit{and} \quad \partial_\lambda\chi_k = \sum_{\ell\neq k}\frac{\langle\chi_k\,\chi_\ell\,\partial_\lambda V\rangle}{\epsilon_k - \epsilon_\ell}\chi_\ell.$$

Using these last two lemmata we can prove (see Appendix of [5]):

Lemma A.3 *Let V and $\widetilde{V}$ be two real-valued functions in $L^2(0,1)$. Then there exist two positive constants C_1 and C_2 independent of k, V and $\widetilde{V}$ such that*

$$|\epsilon_k[V] - \epsilon_k[\widetilde{V}]| \leq C_1\exp(C_2(\|V\|_{L^2(0,1)} + \|\widetilde{V}\|_{L^2(0,1)}))\|V - \widetilde{V}\|_{L^1(0,1)}. \tag{A.2}$$

And,

$$\|\chi_k[V]-\chi_k[\widetilde{V}]\|_{L^\infty(0,1)} \le C_1 \exp(C_2(\|V\|_{L^2(0,1)}+\|\widetilde{V}\|_{L^2(0,1)}))\|V-\widetilde{V}\|_{L^1(0,1)}. \tag{A.3}$$

Lemma A.4 *Let* $V\in L^2(0,1)$ *such that* $V\ge 0$, *then the eigenvectors of the Schrödinger operator satisfy*

$$\|\chi_k[V]\|_{L^\infty(0,1)} \le C(1+\|V\|^{1/2}_{L^2(0,1)}).$$

Proof The result of Lemma 1 Chap. 1 of [28] provides:

$$\chi_k(z)=A_k\sin(\sqrt{2\epsilon_k}z)+2\int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}}V(t)\chi_k(t)\,dt, \tag{A.4}$$

where A_k is a nonnegative constant to be determined. Thanks to the Cauchy-Schwarz inequality, we deduced

$$\left|\int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}}V(t)\chi_k(t)\,dt\right| \le \frac{\int_0^1 V(t)|\chi_k(t)|\,dt}{\sqrt{2\epsilon_k}} \le \frac{\langle|\chi_k|^2V\rangle^{1/2}}{\sqrt{2\epsilon_k}}\|V\|^{1/2}_{L^2(0,1)}.$$

Moreover, from (A.1),

$$\epsilon_k=\frac{1}{2}\langle|\partial_z\chi_k|^2\rangle+\langle|\chi_k|^2V\rangle \ge \langle|\chi_k|^2V\rangle.$$

Thus,

$$\left|\int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}}V(t)\chi_k(t)\,dt\right| \le \frac{1}{\sqrt{2}}\|V\|^{1/2}_{L^2(0,1)}. \tag{A.5}$$

Thus from (A.4) we have for all $z\in[0,1]$

$$|\chi_k(z)|\le A_k+\sqrt{2}\|V\|^{1/2}_{L^2(0,1)}. \tag{A.6}$$

Now, we will use the condition $\|\chi_k\|_{L^2(0,1)}=1$ to bound A_k. If we use the expression of χ_k (A.4) in the identity $\int_0^1\chi_k^2\,dz=1$, we obtain

$$\begin{aligned}1 &\ge A_k^2\int_0^1 \sin\big(\sqrt{2\epsilon_k}z\big)^2\,dz\\ &\quad+4A_k\int_0^1 \sin\big(\sqrt{2\epsilon_k}z\big)\int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}}V(t)\chi_k(t)\,dt\,dz.\end{aligned} \tag{A.7}$$

For the second term we have from (A.5)

$$\left|\int_0^1 \sin\big(\sqrt{2\epsilon_k}z\big)\int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}}V(t)\chi_k(t)\,dt\,dz\right| \le \frac{1}{\sqrt{2}}\|V\|^{1/2}_{L^2(0,1)}.$$

And we can calculate

$$\int_0^1 \big[\sin\big(\sqrt{2\epsilon_k}z\big)\big]^2\,dz=\frac{1}{2}-\frac{\sin(2\sqrt{2\epsilon_k})}{4\sqrt{2\epsilon_k}}.$$

We have assumed that $V \geq 0$. It implies $\epsilon_k[V] \geq \epsilon[0] = \frac{1}{2}\pi^2 k^2$, for all $k \geq 1$. Thus we can inject these remarks in (A.7), it leads to

$$1 \geq A_k^2\left(\frac{1}{2} - \frac{1}{4\pi}\right) - 2\sqrt{2}\, A_k \|V\|_{L^2(0,1)}^{1/2}.$$

This implies that there exists a nonnegative constant C such that

$$A_k \leq C(1 + \|V\|_{L^2(0,1)}^{1/2}), \quad \forall k \geq 1.$$

It remains to inject this last estimate in (A.6) to conclude the proof. □

Lemma A.5 *Let V and $\widetilde{V}$ be two given nonnegative potentials in $L^2(0,1)$. Then there exists a nonnegative constant C such that*

$$|\epsilon_k[V] - \epsilon_k[\widetilde{V}]| \leq C(1 + \|V\|_{L_z^2(0,1)}^{1/2} + \|\widetilde{V}\|_{L_z^2(0,1)}^{1/2})\|V - \widetilde{V}\|_{L_z^2(0,1)}. \tag{A.8}$$

Proof This is an easy consequence of Lemmas A.4 and A.2. Indeed, if we denote for $\lambda \in [0,1]$, $W(\lambda, z) = \widetilde{V} + \lambda(V - \widetilde{V})$ and $\epsilon_k(\lambda) = \epsilon_k[W(\lambda,\cdot)]$, we have

$$\epsilon_k[V] - \epsilon_k[\widetilde{V}] = \int_0^1 \partial_\lambda \epsilon_k(\lambda)\, d\lambda = \int_0^1 \langle |\chi_k[W(\lambda,\cdot)](z)|^2 (V - \widetilde{V})\rangle\, d\lambda.$$

Thus, we have

$$|\epsilon_k[V] - \epsilon_k[\widetilde{V}]| \leq \|V - \widetilde{V}\|_{L^2(0,1)} \int_0^1 \|\chi_k[W(\lambda,\cdot)]\|_{L^4(0,1)}^2\, d\lambda.$$

The estimate (A.8) follows then from Lemma A.4 and the interpolation:

$$\|\chi_k[W(\lambda,\cdot)]\|_{L^4(0,1)}^2 \leq \|\chi_k[W(\lambda,\cdot)]\|_{L^2(0,1)} \|\chi_k[W(\lambda,\cdot)]\|_{L^\infty(0,1)}.$$ □

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